Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XVII

Def. A Lie group G is a group endowed with a smooth manifold structure such that the map

$$G \times G \longrightarrow G$$
$$(g, h) \longmapsto g \cdot h^{-1}$$

is smooth [The Cartesian product $M \times N$ of two differentiable manifolds M and N has a natural differentiable manifold structure...]. The above condition is equivalent to requiring the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ to be smooth.

Examples: $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ (general linear groups), $O(n)$ (orthogonal groups), $U(n)$ (unitary groups), $SO(n)$ (special orthogonal groups), $SU(n)$ (special unitary groups ($\det = +1$)) et cetera, are Lie groups. The group operation is just matrix product. Local charts can be constructed by means of the exponential map (see below).

The left and right translations

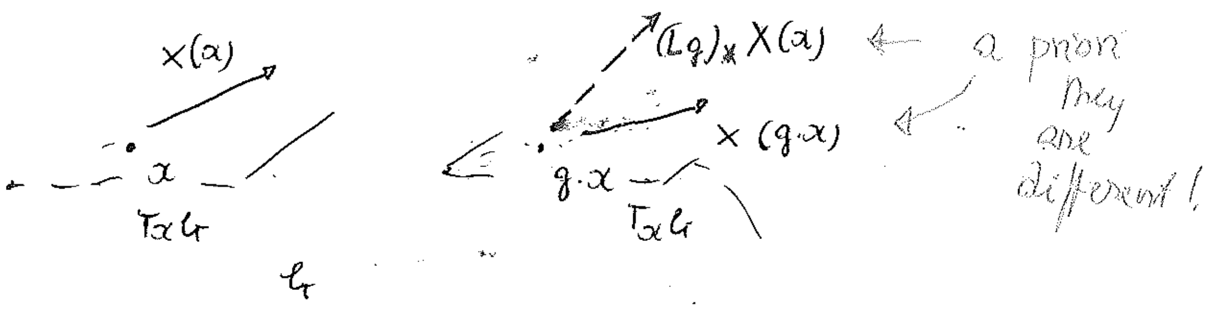
$$L_g : G \ni x \longmapsto g \cdot x \in G$$

$$R_g : G \ni x \longmapsto x \cdot g \in G$$

are diffeomorphisms

Def. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if

$$(\diamond) \qquad X(g\cdot x) = (L_g)_* X(x)$$



and this amounts to the condition:

$$X(g) = (L_g)_* \underset{T_eG}{\underset{\cap}{X(e)}}$$

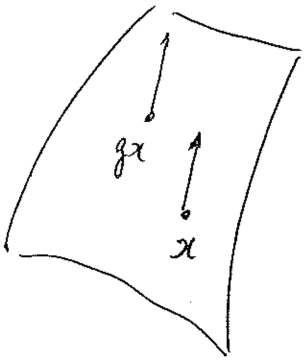
← identity, or neutral element, of G

Notice that $(\diamond)$ can be reformulated, suggestively, as

$$(\diamond\diamond) \qquad (L_g)_* X = X$$

$y = g\cdot x$

$x = g^{-1}\cdot y$



Indeed

$$((L_g)_* X)(\underset{G}{\underset{\cap}{y}}) \underset{\text{def}}{=} (L_g)_* (X(g^{-1}\cdot y))$$

$$\underset{\text{left invariance}}{=} X(g\cdot g^{-1}\cdot y) = X(y), \text{ which is } (\diamond\diamond).$$

Also, since L_g is a diffeomorphism, one has for l. inv. X, Y:

$$(L_g)_*([X,Y]) = [(L_g)_* X, (L_g)_* Y] = [X,Y],$$

that is, $[X,Y]$ is also left-invariant.

Similarly, one defines right invariant v. fields

This entails that
$$\mathfrak{g} \equiv \{ \text{left invariant v. fields of } G \}$$
is a Lie algebra (with respect to $[\ ]$), termed

Lie algebra of G (older, and possibly better terminology, infinitesimal Lie group associated to G)

(it is actually a Lie subalgebra of all $\mathfrak{X}(G)$)

| every Lie algebra is the Lie algebra of a Lie group (Lie's theorem). We shall not prove this result. |

Notice that, as vector spaces, $\mathfrak{g} \cong T_e G$ and $\dim \mathfrak{g} = \dim G$

$T_e G$: tangent space to G at e.

Let us consider the flow of $X \in \mathfrak{g}$. One has the following

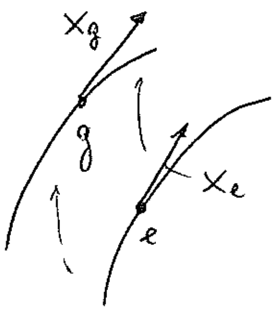
★ Theorem Let $X \in \mathfrak{g}$. Then

(a) $$F_t^X(g) = g \cdot F_t^X(e) \equiv L_g \cdot F_t^X(e)$$

i.e. the integral curves of X are obtained simply by translating the integral curve passing through e.

(b). X is complete, i.e its flow F_t^X is defined $\forall t \in \mathbb{R}$

Proof. Ad (a). For $t \in I$ (a suitable interval $\ni 0$), the two curves



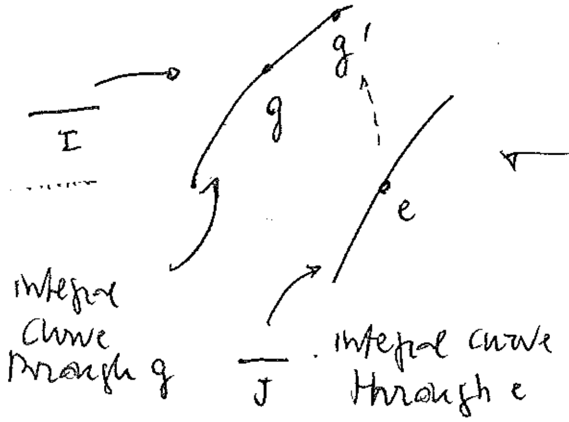
$$t \longmapsto F_t^X(g)$$
$$t \longmapsto g \cdot F_t^X(e)$$

recall $(\psi \circ \varphi)_* = \psi_* \cdot \varphi_*$ (chain rule)

both pass through g (for $t=0$).

Let us compute their velocities at g.
$$\left.\frac{d\ F_t^X(g)}{dt}\right|_{t=0} = X_g \ ; \quad \left.\frac{d(g \cdot F_t^X(e))}{dt}\right|_{t=0} = (L_g)_* X_e$$
$= X_g$ (by left invariance), hence they coincide.

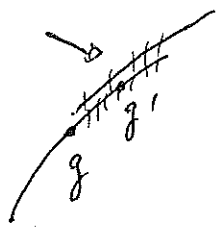
Ad (b). As for the second property, resort to the following "pictorial" argument:



Let us transport over the former curve with e going to g'.

We obtain an integral curve defined on a bigger interval.

They coincide by part (a)

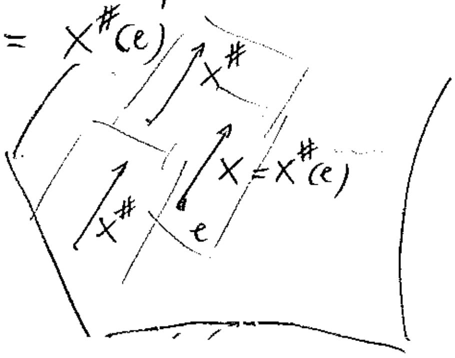


Therefore, if I is <u>maximal</u>, then it must coincide with $\mathbb{R}$ □.

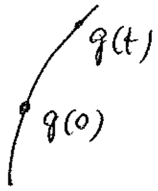
Let us write (a) with a different notation

$\mathfrak{g} \cong T_e G \ni X$. $X^\# =$ left inv. vector field induced by $X = X^\#(e)$

$$X^\#(g) = (L_g)_* \underbrace{X^\#(e)}_{= X}$$



Integral curves: $g = g(t)$



Then:

$$\boxed{\dot{g}(t) = (L_{g(t)})_* X} \quad (\maltese)$$

velocity of g(t)

In particular, if G is a matrix group, (*) becomes

$$\dot{g} = g \cdot X$$ matrix multiplication

$$\Rightarrow \quad g(t) = g(0)\, e^{tX}$$

$e^{Y} = \sum_{k=0}^{\infty} \frac{Y^k}{k!}$, convergent $\forall t \in \mathbb{R}$
matrix exponential (any norm on M_n)

In fact, if $\gamma = \gamma(t)$ is a smooth curve in G, a matrix group, with $\dot{\gamma}(0) = \Xi$ (a matrix), we have

$$\frac{d}{dt}\left(g \cdot \gamma(t)\right)\Big|_{t=0} = g \cdot \dot{\gamma}(0) = g \cdot \Xi$$

fixed

matrix product

We then set: $\mathbb{R} \ni t \longmapsto F_t^X(e) =: \exp(tX)$

integral curve of $X \in \mathfrak{g}$ through e

and we call it 1-parameter group generated by X.

$$\exp_X : \begin{array}{l} \mathbb{R} \longrightarrow G \\ t \longmapsto \exp(tX) = F_t^X(e) \end{array}$$

is indeed a group homomorphism and $\{\exp(tX)\}_{t \in \mathbb{R}} = \exp_X(\mathbb{R})$ becomes an abelian subgroup of G

⚠ Notice: for matrices $e^{tX} e^{sX} = e^{(t+s)X} = e^{sX} e^{tX}$
but in general $e^X e^Y \neq e^{X+Y} \neq e^Y e^X$ ⇝ CBH Formula

The map

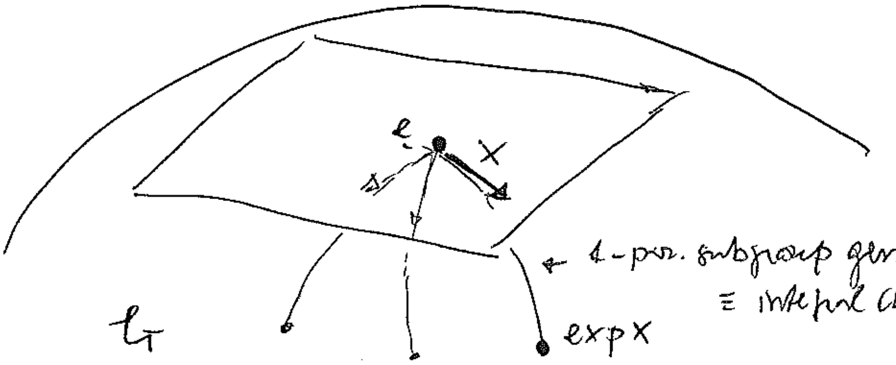
$$\exp : \mathfrak{g} \longrightarrow G$$
$$X \longmapsto \exp X = F_1^X(e)$$

is called exponential map. Locally, it is a diffeomorphism since

$$\exp_*\big|_0 = I_n \qquad \left(\frac{d \exp tX}{dt}\bigg|_{t=0} = X \right)$$

(by the inverse function theorem, which holds on manifolds)

think of the matrix group case



← 1-par. subgroup gen. by X $\equiv$ integral curve of $X^\#$ through e

⚠ The exponential map can be computed explicitly at any point, but its expression is quite <u>clumsy</u> (it involves the so-called Campbell-Baker-Hausdorff formula).

Examples

1. $\mathbb{R}^n$ $\quad (x, y) \longmapsto x + y$ $\quad$ (abelian group)

$$L_x y = x + y = R_x y$$

Let $x' = a + x$ $\quad dx' = dx$ $\quad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$

$$(L_a)_* = I$$

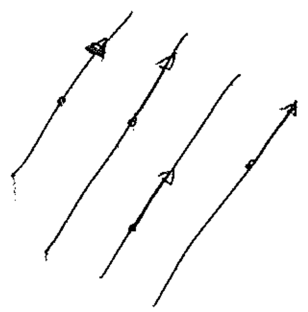
$$(L_a)_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$$

I claim that $\quad Lie(\mathbb{R}^n) = \mathbb{R}^n = \{\text{constant vector fields}\}$.

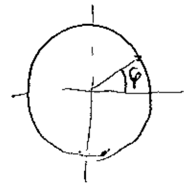
$$(L_a)_* \left(b^i(x) \frac{\partial}{\partial x^i} \right) = b^i(x+a) \frac{\partial}{\partial x^i} = b^i(x) \frac{\partial}{\partial x^i}.$$

$\iff b^i(x+a) = b^i(x)$ $\quad \forall x \in \mathbb{R}^n$, $\forall a \in \mathbb{R}^n$, i.e. $b^i(x) \equiv b^i$ constant

Integral curves: straight lines (translates of lines through the origin)



2. The circle S^1 $\qquad$ $Lie(S^1) = \mathbb{R}\frac{\partial}{\partial\varphi}$

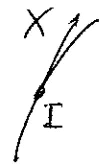


2'. The torus $\mathbb{T}^n = \underbrace{S^1 \times \dots S^1}_{n}$ $\qquad$ $Lie(\mathbb{T}^n) = \mathbb{R}^n$

3. $G = GL_n(\mathbb{R})$. $\qquad$ $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ all matrices $\qquad$ $[\,,\,]$ = matrix commutator

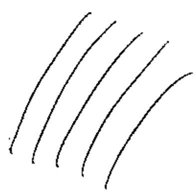
In fact every $X \in \mathfrak{g}$ generates the following 1-parameter group

$$g(t) = e^{tX} \quad (\equiv \exp(tX))$$



whose translates (left) $\tilde{g}(t) = g_0 \cdot e^{tX}$

yield the integral curves of $X^\#$, the l. inv. vector field corresponding to X



Notice that, for $A \in G$, near I (use any norm), then $A = e^X$ for a unique $X \in \mathfrak{g}$:

indeed, set $A = I + K$, with $\|K\| < 1$

then $A = \log(I+K) = K - \frac{K^2}{2} + \frac{K^3}{3} + \dots$ $\qquad$ (†) (convergent)

⚠ In order to complete the identification, one should prove that

$$\underset{\text{Lie bracket}}{[X^\#, Y^\#]} = \underset{\text{matrix commutator}}{[X, Y]}^\#$$

This can be seen as follows

Let

$$X^{\#}\Big|_{A \in Gl(n,\mathbb{R})} = A\cdot X = A\cdot X^{\#}\Big|_{I}$$

$X^{\#}$ corresponding to $X \in gl(n,\mathbb{R})$

actually

$$X^{\#}\Big|_{A} = \underbrace{A^{i}_{\ j} \overset{X:\ \text{matrix}}{X^{j}_{\ k}}}_{\xi^{i}_{\ k}} \frac{\partial}{\partial A^{i}_{\ k}} \qquad \text{akin to: } \left(\xi^{I} \frac{\partial}{\partial A^{I}} \right)$$

$$Y^{\#}\Big|_{A} = \underbrace{A^{i}_{\ j} Y^{j}_{\ k}}_{\eta^{i}_{\ k}} \frac{\partial}{\partial A^{i}_{\ k}} \qquad \frac{\partial A^{i}_{\ r}}{\partial A^{j}_{\ l}} = \delta^{ij}_{rl}$$

Then

$$[X^{\#}, Y^{\#}]\Big|_{I_n} = \dots = \underbrace{(X^{i}_{\ k} Y^{k}_{\ r} - Y^{i}_{\ k} X^{k}_{\ r})}_{[X,Y]^{i}_{\ r}} \frac{\partial}{\partial A^{i}_{\ r}}\Big|_{I_n}$$

$$\Rightarrow [X^{\#}, Y^{\#}] = [X,Y]^{\#} \quad \text{(at all points)}$$

as claimed

(†) continues from preceding page

This illustrates the basic property of the exponential map of being a local diffeomorphism between suitable neighbourhoods of $0 \in \mathfrak{g}$ and $e \in G$, respectively

4. $G = Gl_n(\mathbb{C})$ (viewed as a real group)

idem $\mathfrak{g} = M_n(\mathbb{C})$

5. $U(n)$ unitary group $= \{ U \in Gl_n(\mathbb{C}) \,/\, U^*U = UU^* = I_n \}$

$$U^* = \overline{U^T} = \overline{U}^T \qquad U^{-1} = U^*$$

(linear transformation leaving the standard hermitian inner product invariant)

$$\mathfrak{g} = \mathfrak{u}(n) = \{ X \in M_n(\mathbb{C}) \,/\, X^* + X = 0 \}$$

Indeed, let $U = U(t)$ a smooth curve passing through I_n (specifically $U(0) = I_n$; $t \in \mathcal{Y}$ interval) with velocity X, for instance $U(t) = e^{tX}$, $X \in M_n(\mathbb{C})$. We require it to lie in $U(n)$.

Therefore $U^*(t)\,U(t) = I \quad \forall t \in \mathcal{Y}$. Differentiating at $t=0$ yields

$$0 = \frac{d}{dt} U^*(t)U(t) = \dot{U}^*(t)\,U(t) + U^*(t)\,\dot{U}(t) \qquad \forall t \in \mathcal{Y}$$

$$\Rightarrow \quad \dot{U}^*(0)\underbrace{U(0)}_{= I_n} + \underbrace{U^*(0)}_{= I_n}\dot{U}(0) = 0 \qquad \dot{U^*} = \dot{U}^*$$

$$X^* \quad + \quad X \quad = 0$$

Slightly differently
$$U = I + tX + o(t)$$
$$U^* = I + tX^* + o(t)$$

$$U^*U = I + t\underbrace{(X + X^*)}_{= 0} + \dots$$

That is, one "imposes $U^*U = I$ at first order".

$X + X^* = 0$ is the infinitesimal version of $U^*U = I$: in fact, anciently, Lie algebras where called "infinitesimal Lie groups" – a possibly better name.

$5'$. $G = SU(n)$ $\quad \mathfrak{g} = \{ X \in \mathfrak{u}(n) \,/\, \operatorname{tr} X = 0 \}$

we have to impose the extra condition $\det U(t) \equiv 1$

$U(t) = I + tX + \ldots \qquad \det U(t) = \det(1 + tX + \ldots)$

$0 = \frac{d}{dt} \det U(t) = \operatorname{tr} X$, either directly or via the argument in $\boxed{}$

$6.$ $G = O(n)$; $\quad \mathfrak{g} = \{ X \in M_n(\mathbb{R}) \,/\, X^T + X = 0 \}$

$6'$. $G = SO(n)$; $\mathfrak{g}$ $\mathbin{/\!/}$

$X^T + X = 0 \implies \operatorname{Tr} X = 0$

Hence $\mathfrak{o}(n) = \mathfrak{so}(n)$

This is not surprising, since $SO(n)$ is the connected component of $O(n)$ containing I_n, so their Lie algebras ($\cong$ tangent spaces at the identity) must coincide.

$\det e^A = e^{\operatorname{Tr} A}$
true for diagonal matrices, then for diagonalizable matrices, via Spectral Theorem, then for all matrices, via density. Actually, unitary matrices are diagonalizable

Or, one uses the fact that complex matrices can be cast into a triangular form

* On the interpretation of $[X, Y]$
(Special case)

take $G = Gl_+(n, \mathbb{R})$ $\qquad e = I_n$

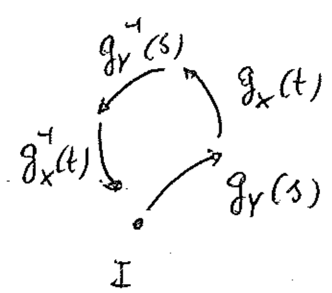
$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$

$[\,,\,] =$ matrix commutator

let $g_X(t) = e^{tX}$ the 1-parameter group generated by $X \in \mathfrak{g}$. Let us evaluate

$$g_X^{-1}(t) \cdot g_Y^{-1}(s) \cdot g_X(t) \circ g_Y(s)$$

$$\left(= (g_Y \circ g_X)^{-1} g_X g_Y \right)$$



keep these terms

$$g_X \cdot g_Y = (I + tX + \dots)(I + sY + \dots)$$
$$= I + tX + sY + stXY + \dots$$

$$g_Y \cdot g_X = I + \underbrace{tX + sY + stYX}_{\xi} + \dots$$

$(1+\xi)^{-1} = 1 - \xi + \xi^2 + \dots$

geometric series

$$(g_Y \cdot g_X)^{-1} = I - tX - sY - stYX$$
$$+ stXY + stYX + \dots$$
$$= I - tX - sY + stXY$$

$I - tX - sY + stXY$

$I + tX + sY + stXY$

$\Rightarrow (g_Y g_X)^{-1} g_X g_Y =$

$$I + stXY + stXY - stXY - stYX + \dots$$
$$= I + st[X, Y] + \dots$$

Thus $$\frac{\partial^2}{\partial s \partial t} \left(g_X^{-1}(t)\, g_Y^{-1}(s)\, g_X(t)\, g_Y(s) \right) \Big|_{\substack{t=0\\ s=0}} = [X, Y]$$

(slightly differently, work with $s \to \sqrt{s}$, $t \to \sqrt{s}$

$$\ldots = I + s[X, Y] + \ldots \qquad \frac{\partial}{\partial s}(\;)\Big|_{s=0} = [X, Y]$$

recall $[X, Y]_{Lie} = [X, Y]_{matrix}$