— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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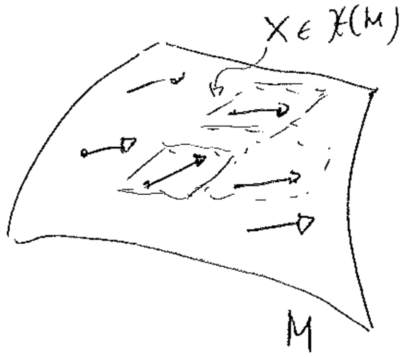
Lecture XV



Let $X, Y \in \mathcal{X}(M)$ (vector fields on M)

The Lie bracket of X and Y, denoted by $[X, Y]$, is the vector field defined via the following commutator

$$\underbrace{[X,Y](f)}_{\mathcal{C}^\infty(M)} := X(\underbrace{Y(f)}_{\mathcal{C}^\infty(M)}) - Y(X(f))$$

So one can apply X to it

We must verify that indeed we get a vector field. This can be ascertained via a local coordinate computation

$$X = a^i \frac{\partial}{\partial x^i} \quad , \quad Y = b^j \frac{\partial}{\partial x^j}$$

Einstein convention

$$X(Y(f)) = a^i \frac{\partial}{\partial x^i}\left(b^j \frac{\partial f}{\partial x^j}\right) = a^i \frac{\partial b^j}{\partial x^i}\frac{\partial f}{\partial x^j} + a^i b^j \frac{\partial^2 f}{\partial x^j \partial x^i}$$

$$Y(X(f)) = b^j \frac{\partial}{\partial x^j}\left(a^i \frac{\partial f}{\partial x^i}\right) = b^j \frac{\partial a^i}{\partial x^j}\frac{\partial f}{\partial x^i} + a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j}$$

equal by Schwarz

$$\Rightarrow \quad X(Y(f)) - Y(X(f))$$

exchange indices $i \leftrightarrow j$

$$= a^i \frac{\partial b^j}{\partial x^i}\frac{\partial f}{\partial x^j} - b^i \frac{\partial a^j}{\partial x^i}\frac{\partial f}{\partial x^j} = \longrightarrow$$

$$[X,Y](f) = \underbrace{\left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}\right)}_{c_j} \frac{\partial f}{\partial x^j}$$

or:

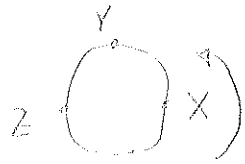
$$[X,Y] = \underbrace{\left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}\right)}_{c_j} \frac{\partial}{\partial x^j}$$

which is indeed a vector field.

It is readily checked that $[\,,\,]$ fulfils the following properties:

1. $[\,,\,]$ is bilinear
2. $[\,,\,]$ is skew symmetric ($[Y,X] = -[X,Y]$) $\forall X,Y \in \mathfrak{X}(M)$
3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ $\quad \forall X,Y,Z \in \mathfrak{X}(M)$

(Jacobi identity) cyclical permutations



Namely, $(\mathfrak{X}(M), [\,,\,])$ is a Lie algebra (over $\mathbb{R}$), see below (its dimension, as a vector space, is infinite)

Also recall that $\mathfrak{X}(M)$ is a $C^\infty(M)$-module

($X \in \mathfrak{X}(M) \Rightarrow fX \in \mathfrak{X}(M)$ (+ other properties...)

$(fX)(g) := f \cdot X(g)$ pointwise product $\longrightarrow$ see next page

We also notice that if $\psi : M \to N$ is a diffeomorphism, then $\psi_*([X,Y]) = [\psi_* X, \psi_* Y]$ (either via a local coordinate calculation or via: $\psi_*([X,Y])(f)(y) = [X,Y](\psi^* f)(\psi^{-1}(y))$

which is easily see to be equal to the l.h.s. $\longrightarrow$ see next page

Reminder

A ring $(A, +, \circ)$ (or simply A, if no confusion arises)

is an abelian group w.r. to $+$, $\circ$ is an associative multiplication and these operations are distributive:

$$a(b+c) = ab + ac \qquad \forall\, a,b,c \in A$$
$$(b+c)a = ba + ca$$

An abelian group M is called A-module if A acts "linearly" on it, namely:

There exists a map $\mu : A \times M \to M$
$$(a,x) \mapsto \mu(a,x) \equiv a \cdot x$$
($a \cdot x$ shortly)

such that:

$$a(x+y) = ax + ay$$
$$(a+b)x = ax + bx$$
$$(ab)x = a(bx)$$
$$1 \cdot x = x$$

For instance, a vector space is a K-module (K a field)

$\psi_*[X,Y] = [\psi_* X, \psi_* Y]$ (continued)

$$\psi_* X((\psi_* Y)(f))(y) = X(\psi^*[(\psi_* Y)(f)])(\psi^{-1}(y))$$

Crucial step
$$= X(Y(\psi^* f))(\psi^{-1}(y)) = X \cdot Y(\psi^* f)(\psi^{-1}(y))$$

therefore, collecting terms, we have r.h.s. =

$$(X \cdot Y - Y X)(\psi^* f)(\psi^{-1}(y)) = \text{l.h.s.}$$

$$\psi^*[(\psi_* Y)(f)](\psi^{-1}(y)) = (\psi_* Y)(f)(\psi \circ \psi^{-1}(y)) = \psi_*(Y)(f)(y)$$
$$= Y(\psi^* f)(\psi^{-1}(y))$$

$(\psi^* g)(x) = g \circ \psi(x)$

* Question : does the map

$$(X,Y) \longmapsto [X,Y] = \tau(X,Y)$$

define a tensor (of type $(1,2)$) ?

" feed τ with 2 vector fields, produce a vector field (type $(1,0)$)"

NO! $[\alpha X, \beta Y] \neq \alpha\beta [X,Y]$ ⚠

$\alpha, \beta \in \mathcal{C}^\infty(M)$

In fact:

$$[\alpha X, \beta Y](f) = \alpha X\big((\beta Y)(f)\big) - \beta Y\big((\alpha X)(f)\big)$$

$$= \alpha\big(X(\beta)Y(f) + \beta XY(f)\big) - \beta\big(Y(\alpha)X(f) + \alpha YX(f)\big)$$

$$= \underbrace{\alpha\beta\,[X,Y](f)}_{\text{ok}} + \{\alpha X(\beta) Y - \beta Y(\alpha) X\}(f)$$

" non tensorial piece"

you just have multilinearity over constants..

★ Digression: Lie algebras

Def. A Lie algebra $(L, [\,,\,])$ (over a field K) is a vector space L over K equipped with a map (Lie bracket)

$$[\,,\,] : L \times L \longrightarrow L$$
$$(X, Y) \longmapsto [X,Y]$$

fulfilling

1. bilinearity
2. Skew-symmetry
3. Jacobi identity

Examples (with $K = \mathbb{R}$)

1. $M_n(\mathbb{R})$ (square matrices) $[A,B] := AB - BA$

   matrix product

2. $\mathfrak{so}(n)$ antisymmetric matrices

   Notice that Sym, by contrast, is NOT a Lie algebra

$A^T = -A$, $B = -B^T$ $\Rightarrow ([A,B])^T = (AB - BA)^T =$

$= B^T A^T - A^T B^T = BA - AB = -[A,B]$ □

3. $(\mathbb{R}^3, \times)$ ($\overset{\text{isomorphic}}{\cong} \mathfrak{so}(3)$, as Lie algebras)

   vector product

4. $(\mathfrak{X}(M), [\,,\,])$ Lie bracket for vector fields, defined above

5. Take $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ $q = (q_1 \dots q_n)$ $p = (p_1 \dots p_n)$

braces

$$\{f, g\}(q,p) := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

$f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ Poisson bracket

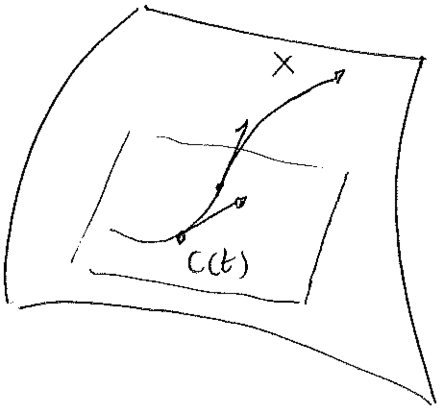
Fundamental in Mechanics!

★ Flow of a vector field

Let $X \in \mathcal{X}(M)$ (vector field on M)

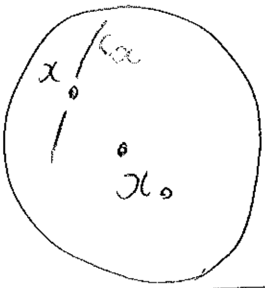
A smooth curve $c = c(t)$ in M, $t \in I$ (some interval, containing 0)

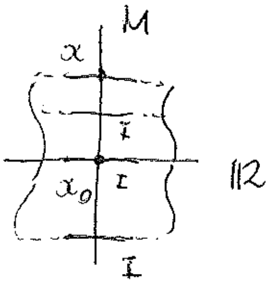
$c : I \subset \mathbb{R} \longrightarrow M$ is called an integral curve of X if $\dot{c}(t) = X(c(t))$, that is, if its velocity at $c(t)$ equals X, evaluated at $c(t)$ (both are vectors in $T_{c(t)}M$)



Without insisting on details, it is clear that an existence & uniqueness theorem, together with smooth dependence on initial conditions (Cauchy-Lipschitz) holds on a smooth manifold as well. More precisely:

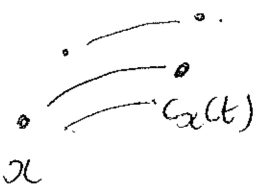
$\forall x_0 \in M$, $\exists\ V \ni x_0$ (neighbourhood), $I \ni 0$ (interval) such that, $\forall x \in V$, $\exists!$ integral curve of X, call it c_x, defined on I, with $c_x(0) = x$, and such that the map $(t, x) \longmapsto c_x(t)$ is smooth.†





The maps

$$x \longmapsto c_x(t) \equiv F^X_t(x)$$



give rise to local diffeomorphisms fulfilling ⟶

† If M_1, M_2 are manifolds, $M_1 \times M_2$ is a manifold as well...

$$(\diamond) \qquad \boxed{F^X_{t_1} \circ F^X_{t_2} = F^X_{t_1+t_2}}$$

group property

whenever the l.h.s and the r.h.s. are both defined

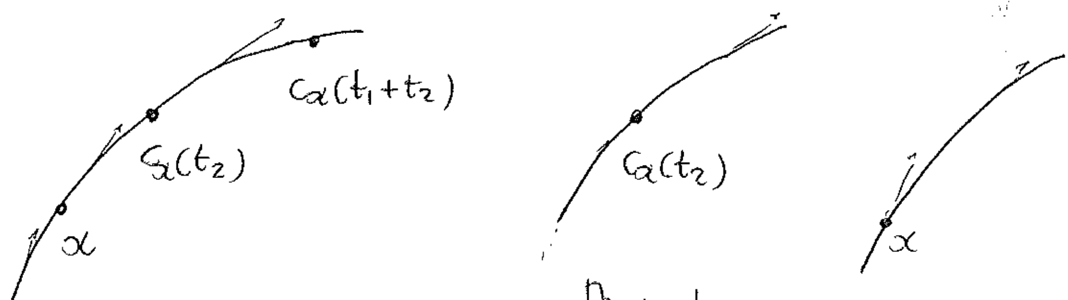
the $\{F^X_t\}$ define a <u>local</u> <u>1-parameter group</u> of <u>local</u> <u>diffeomorphisms</u>

That $(\diamond)$ holds is clear since

$c_x(t_1+t_2)$ is the point of the integral curve, at "time" t_1+t_2, starting from $c_x(0) = x$ at "time" 0, which, by <u>Cauchy-Lipschitz</u>, coincides

$\| \; F^X_{t_1+t_2}(x)$

with the point of the integral curve, at t_1, starting, at time 0, from $c_x(t_2)$



These two curves coincide, since

at t_2 they pass through the same point $(c_x(t_2))$ and have the same velocity (Cauchy-Lipschitz) there.

One says that X generates a local 1-parameter group
(or: X is a generator)

of local diffeomorphisms.

Conversely, given $\{F_t\}_{t\in I}$, local 1-parameter group

of local diffeomorphisms, one defines:

$$X(f)(x) := \lim_{t\to 0} \frac{f(F_t(x)) - \overset{F_0(x)}{\overset{\shortparallel}{f(x)}}}{t}$$

$$= \frac{d}{dt}\psi(t)\Big|_{t=0} \qquad \psi(t) = f(F_t(x)) \quad x \text{ fixed}$$

$$\psi(0) = f(F_0(x)) = f(x)$$

that is, restrict f on the curve
and differentiate at $t=0$

a smooth function in a neighbourhood of x
(it can be extended to a global function vanishing outside
a bigger neighbourhood by means of a suitable
partition of unity)

X: generator of $\{F_t\}$.

Also, ★ Lie derivative of f along X

upon defining $(\mathcal{L}_X f)(x) = \dfrac{d f(F_t^X(x))}{dt}\Big|_{t=0} =$

$= \lim\limits_{t\to 0} \dfrac{f(F_t^X(x)) - f(x)}{t}$, we obviously have $\mathcal{L}_X f = X(f)$

* Examples

1. $M = \mathbb{R}^2$

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$O:(0,0)$ is the only critical point of X (i.e. $X(x,y)=0$ if and only if $(x,y)=(0,0)$)

Let us find its integral curves:

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \qquad (\dot{c}(t) = X(c(t)))$$

We find $\ddot{x} = -\dot{y} = -x \Rightarrow \ddot{x} + x = 0$ (harmonic oscillator)

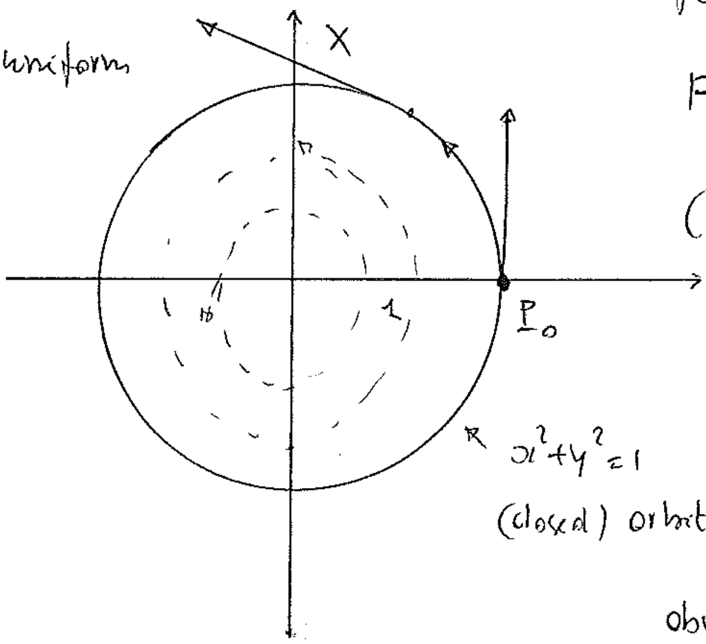
fix $P_0:(x_0,y_0) = (1,0)$. The integral curve passing through it is $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

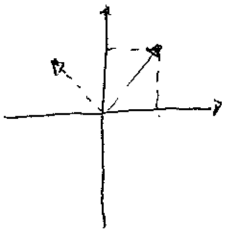
We have a global flow:

$$F_t^X = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(defined $\forall t \in \mathbb{R}$ (one has however periodicity) and $\forall P_0 \in M = \mathbb{R}^2$

X generates uniform rotations around the origin





$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}}_{R_t} \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{P_0 \text{ (any pt)}}$$

rotation around O, of an angle $\varphi = t$

obviously $F^X_{t_1+t_2} = F^X_{t_1} \circ F^X_{t_2}$

$(R_{t_1+t_2} = R_{t_1} \cdot R_{t_2})$

Conversely, starting from $\{R_t\}_{t\in\mathbb{R}}$ (rotation flow)

one computes its generator (it should be $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$)

at $P_0 = (x_0, y_0)$ as follows. Calculate

$$\frac{d}{dt} R_t \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\Bigg|_{t=0} = \frac{d R_t}{dt} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix}\Bigg|_{t=0} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix}, \quad \text{that is, this is } X(P_0)$$

The computation is eased by the fact that in this case $\mathbb{R}^2$ can be identified with the tangent space $T_p\mathbb{R}^2$ at each p.

Let us proceed more formally; we have to compute $\forall f \in C^\infty(\mathbb{R}^2)$

$$\frac{d}{dt} f(R_t \cdot P_0)\Big|_{t=0} = \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)\Bigg|_{t=0}$$

$x = \cos t\, x_0 - \sin t\, y_0$

$y = \sin t\, x_0 + \cos t\, y_0$

$\frac{dx}{dt} = -\sin t\, x_0 - \cos t\, y_0$

$\frac{dy}{dt} = \cos t\, x_0 - \sin t\, y_0$

$\frac{dx}{dt}\Big|_{t=0} = -y_0$

$\frac{dy}{dt}\Big|_{t=0} = x_0$

$$= -y_0 \frac{\partial f}{\partial x}(P_0) + x_0 \frac{\partial f}{\partial y}(P_0)$$

i.e. generator at P_0 =

$$\left(-y_0\frac{\partial}{\partial x} + x_0\frac{\partial}{\partial y}\right)\Bigg|_{P_0}$$

namely X, at P_0

2. On $\mathbb{R}$, consider $X = x^2 \dfrac{\partial}{\partial x}$

integral curves:

$$\dot{x} = x^2 \qquad \frac{dx}{x^2} = dt \quad \text{(separation of variables)}$$

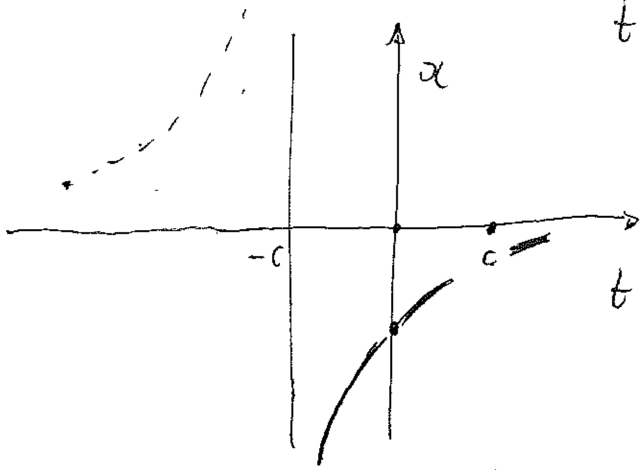
$x \neq 0$

$$\Rightarrow \quad -\frac{1}{x} = t + c$$

$$\Rightarrow \quad x = -\frac{1}{t+c} \qquad t \neq -c$$

branch of a hyperbola

$x(0) = -\dfrac{1}{c}$ — The flow is only local.



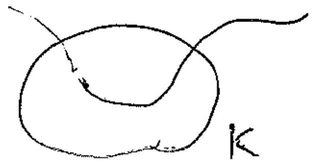
$t + c > 0$

here $c > 0$

$\boxed{\text{The maximal interval is not } \mathbb{R}}$

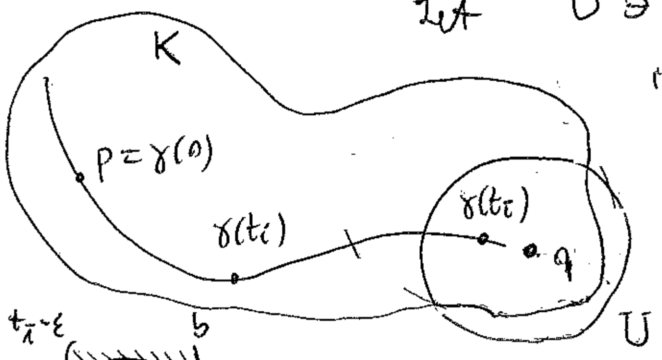
Also observe that $\mathrm{Im}\,(x = x(t)) = (-\infty, 0)$ which is not contained in any compact set in $\mathbb{R}$.
(since a compact set in $\mathbb{R}^n$ is closed and bounded)
This is an instance of a general phenomenon described by the Escape Lemma (see next page)

★ **Escape lemma** (Lemma di fuga). Let $X \in \mathfrak{X}(M)$. If γ is an integral curve of X defined in a maximal domain which is not $\mathbb{R}$, then $\mathrm{Im}\,\gamma$ is not fully contained in any $K \subset M$, K compact (that is, it eventually "escapes" from any K)
[Cf. the previous example]

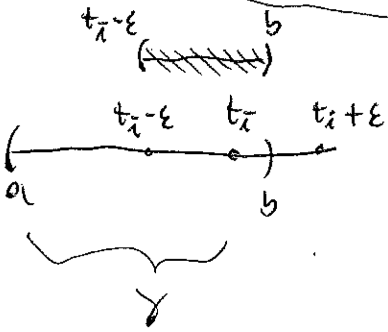


Pf. By contradiction, let $(a,b) \ni t \mapsto \gamma(t) \in M$, (a,b) maximal, and $\mathrm{Im}\,\gamma \subset K$, compact. Let $t_i \to b$. Then $\{\gamma(t_i)\} \subset K$ admits a convergent subsequence, still denoted in the same way, $\gamma(t_i) \to q \in K$.

Let $U \ni q$, $\varepsilon > 0$ s.t. F^X (flow of X) is defined in $(-\varepsilon,\varepsilon) \times \bar{U}$.



Let $\bar{i}$ s.t. $\gamma(t_{\bar{i}}) \in \bar{U}$ and assume $t_{\bar{i}} + \varepsilon > b$. Define the following curve (extending γ):



$$\sigma(t) = \begin{cases} \gamma(t) & t \in (a,b) \\ (F^X_{t - t_{\bar{i}}} \circ F^X_{t_{\bar{i}}})(p) & , t \in (t_{\bar{i}} - \varepsilon, t_{\bar{i}} + \varepsilon) \end{cases}$$

Then (Cauchy-Lipschitz) $\sigma = \gamma$ on $(t_{\bar{i}} - \varepsilon, b)$, σ extends γ, this contradicting maximality of (a,b). □

Corollary. If M is compact, every $X \in \mathfrak{X}(M)$ is complete, i.e. its flow $\{F^X_t\}$ is defined $\forall t \in \mathbb{R}$.
Pf. Trivial.