Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture III

Tensor product spaces p. 1
(p,q) tensors p. 4
Examples p. 5

* Tensor products

Let V, W (finite dimensional) vector spaces over a field K ($\dim_K V = n$, $\dim_K W = m$), their tensor product $V \otimes_K W$ (or $V \otimes W$, if no confusion arises) is defined as follows

$$\boxed{V \otimes_K W := \{ \pi : V^* \times W^* \longrightarrow K \;/\; \pi \text{ bilinear} \}}$$

It is naturally a vector space, generated by π's of the form

$$v \otimes w = \pi : (\underset{V^*}{x^*}, \underset{W^*}{y^*}) \longmapsto \underbrace{x^*(\overset{V}{v})}_{K} \cdot \underbrace{y^*(\overset{W}{w})}_{K}$$

decomposable vectors in $V \otimes W$

product in K

Notice that $\alpha (v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$

$\forall \alpha \in K$, $v \in V$, $w \in W$, and

$$(\alpha v_1 + \beta v_2) \otimes w = \alpha (v_1 \otimes w) + \beta (v_2 \otimes w) \quad \text{etc.}$$

(as maps).

Given, as usual, bases $e = (e_1 \dots e_n)$, $f = (f_1 \dots f_m)$ in V and W, resp., it is easily checked that

$e \otimes f \equiv (e_i \otimes f_j)_{\substack{i=1..n \\ j=1..m}}$ yields a basis for $V \otimes W$, i.e, any $\pi \in V \otimes W$ can be written as

$$\pi = \sum_{i,j} c_{ij}\, e_i \otimes f_j \text{, for uniquely determined } c_{ij} \in K.$$

Hence $\dim V \otimes W = \dim V \cdot \dim W = n \cdot m$

Notice, in particular, $(e_i \otimes f_j)(e^*_k, f^*_\ell) =$

$$= e^*_k(e_i) \cdot f^*_\ell(f_j) = \delta_{ik}\,\delta_{\ell j}$$

and that, if $v = \sum_i \alpha_i e_i$, $w = \sum \beta_j f_j$,

then $$v \otimes w = \underbrace{\sum_{i,j} \alpha_i \beta_j}_{c_{ij}}\, e_i \otimes f_j$$

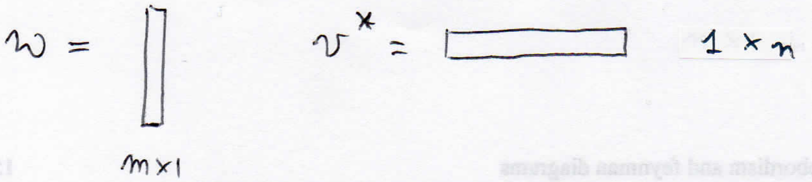
Important remark

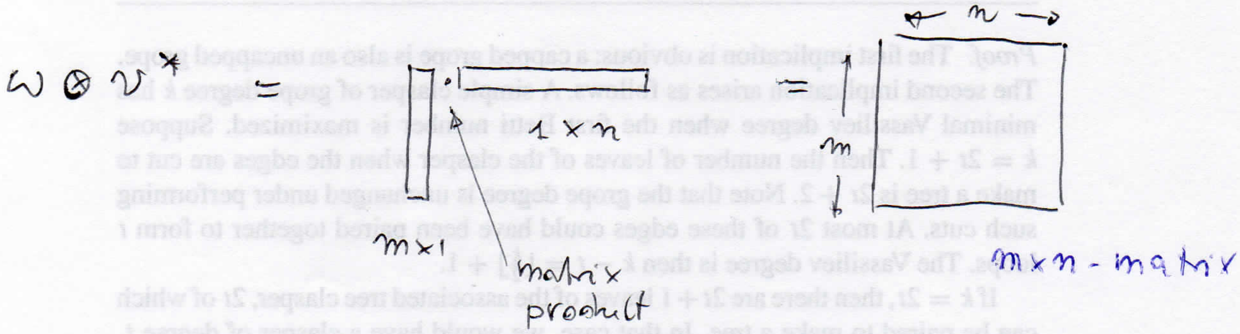
$$\operatorname{Hom}(V, W) \underset{\text{canonically}}{\cong} W \otimes V^*$$

This comes from $$(w \otimes v^*)(\underset{\underset{V}{\cap}}{x}) = \underbrace{\overset{\overset{K}{\cup}}{v^*(x)}}\, \overset{\overset{W}{\cup}}{w}$$

leave it as it stands

Concretely, choosing bases

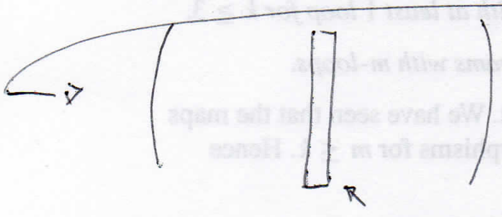




$$\boxed{W \otimes V^* = \left\{ \sum_{i,j} a_{ij} (f_i \otimes e_j^*) \right\}}$$

$$\sum_{i,j} a_{ij} (f_i \otimes e_j^*)(\cdot, e_k) = \sum_{i,j} a_{ij} \underbrace{e_j^*(e_k)}_{\delta_{jk}} f_i$$

$$= \sum_i a_{ik} f_i \quad \text{i.e.} \quad e_k \longrightarrow \sum_i a_{ik} f_i$$



k-column

of the matrix representation
of a homomorphism

One can naturally define $V_1 \otimes V_2 \otimes V_3$ etc.

either directly or via $(V_1 \otimes V_2) \otimes V_3$

(canonically isomorphic to $V_1 \otimes (V_2 \otimes V_3)$) and concluding by induction.

We shall stick to the case in which $V_i = V$ or V^*

Define the space of (p,q)-tensors, $\boxed{\mathcal{T}_{p,q} = \underbrace{V^* \otimes \dots \otimes V^*}_{q} \otimes \underbrace{V \otimes \dots V}_{p}}$

contravariance index

covariance index

notice

notice

$$\boxed{\mathcal{T}_{p,q} = \left\{ \pi : \underbrace{V \times V \times \dots V}_{q} \times \underbrace{V^* \times \dots V^*}_{p} \to K \;\middle|\; \pi \text{ multilinear, i.e. linear in each argument} \right\}}$$

$V \otimes V^* \cong V^* \otimes V$ canonically

$V \otimes V \otimes V^* \overset{can}{\cong} V^* \otimes V \otimes V \overset{can}{\cong} V \otimes V^* \otimes V$ etc.

$T \in \mathcal{T}_{p,q}$ is p-times contravariant, q-times covariant

Fixing a basis $e = (e_1 \dots e_n)$ in V, together with the dual basis $e^* = (e_1^* \dots e_n^*)$, we have

$$\boxed{T = \sum_{I,J} T^{i_1 \dots i_p}_{j_1 \dots j_q} \overset{e^{j_1} \otimes e^{j_2} \otimes e^{j_q}}{\overset{\shortparallel}{e^*_{j_1} \otimes e^*_{j_2} \dots \otimes e^*_{j_q}}} \otimes e_{i_1} \otimes \dots \otimes e_{i_p}}$$

$I = (i_1 \dots i_p)$ p-multindex..

$J = (j_1 \dots j_q)$ q-multindex

components of T with respect to the basis $(e^{j_1} \otimes e^{j_2} \dots \otimes e^{j_q} \otimes e_{i_1} \dots \otimes e_{i_p})$

Schematically: $\boxed{T = T^I_J \, e^J \otimes e_I}$ notice this

Einstein's convention

⚠ $e^*_j \mapsto e^j$ this notation would be more appropriate

★ Examples V, V^*, $\underset{\text{endomorphisms}}{\mathrm{End}(V)} \cong V \otimes V^* \cong V^* \otimes V$

are easily recovered.

$\boxed{V = \tau_{1,0}}$, $\boxed{V^* = \tau_{0,1}}$

$\boxed{\mathrm{End}(V) \cong \tau_{1,1}}$ notice this i: row index

$\wr$

$T = (a^i_j)$ notice this

$v = x^i e_i$ Einstein's conv.

a^i_j ← row index ← column index

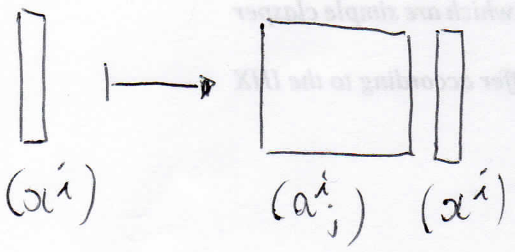
$T = a^i_j e^j \otimes e_i$

★ Compute Tv, using tensor notation (Einstein's convention being employed throughout)
Einstein

$$(a^i_j e^j \otimes e_i)(x^k e_k) = a^i_j x^k \underbrace{e^j(e_k)}_{\delta_{jk}} e_i = a^i_j x^k \delta_{jk} \cdot e_i$$

Einstein

$$= a^i_j x^j e_i \Rightarrow (x^i) \longmapsto (a^i_j x^j)$$



* Another example (crucial in Riemannian geometry)

$\langle \, , \rangle$ inner product on a Euclidean vector space V

$\langle \, \rangle$ is a $(0,2)$-tensor $\langle \, \rangle : V \times V \to \mathbb{R}$ bilinear

(+ symmetric + positive definite)

Upon choosing a basis $e = (e_1 \ldots e_n)$

$v = x^i e_i$ $\langle e_i, e_j \rangle = g_{ij}$

$w = y^j e_j$

$\langle v, w \rangle = g_{ij} x^i y^j$