First-order semantics

$\mathscr{L}$-*structure* (or $\mathscr{L}$-*interpretation*) $\mathfrak{U}$

(1) A non-empty class U, called the *universe of discourse* (or, briefly, *universe*, or *domain*) of $\mathfrak{U}$. The members of U are called *individuals* (of $\mathfrak{U}$).

(2) A mapping which assigns to each function symbol $\mathbf{f}$ of $\mathscr{L}$ an operation $\mathbf{f}^{\mathfrak{U}}$ on U such that if $\mathbf{f}$ is an n-ary function symbol, $\mathbf{f}^{\mathfrak{U}}$ is an n-ary operation on U. In particular, if $\mathbf{a}$ is a constant, then $\mathbf{a}^{\mathfrak{U}}$ is an individual. An individual of this kind — i.e., $\mathbf{a}^{\mathfrak{U}}$ for some constant $\mathbf{a}$ of $\mathscr{L}$ — is said to be *designated*.

(3) A mapping which assigns to each extralogical predicate symbol $\mathbf{P}$ of $\mathscr{L}$ a relation $\mathbf{P}^{\mathfrak{U}}$ on U such that if $\mathbf{P}$ is an n-ary predcate symbol, $\mathbf{P}^{\mathfrak{U}}$ is an n-ary relation on U. In particular, if $\mathbf{P}$ is unary, then $\mathbf{P}^{\mathfrak{U}}$ is a subclass of U.

We therefore define an $\mathscr{L}$*-valuation* σ to be an $\mathscr{L}$-structure $\mathfrak{U}$ together with an assignment of a value $\mathbf{x}^\sigma \in U$ to each variable $\mathbf{x}$.

$$\sigma: VAR \to U$$

By definition, each valuation σ involves a particular structure $\mathfrak{U}$. We refer to $\mathfrak{U}$ as the structure *underlying* σ. If $\mathfrak{U}$ is the structure underlying σ, we define $\mathbf{f}^\sigma$ and $\mathbf{P}^\sigma$ to be the operation $\mathbf{f}^\mathfrak{U}$ and the relation $\mathbf{P}^\mathfrak{U}$, respectively, where $\mathbf{f}$ is any function symbol and $\mathbf{P}$ is any extralogical predicate symbol. The universe U of $\mathfrak{U}$ will also be called the *universe of* σ.

We shall say that two valuations σ and τ *agree* on a given variable $\mathbf{x}$ (or function symbol $\mathbf{f}$, or extralogical predicate symbol $\mathbf{P}$) if σ and τ have the same universe and $\mathbf{x}^\sigma = \mathbf{x}^\tau$ (or $\mathbf{f}^\sigma = \mathbf{f}^\tau$, or $\mathbf{P}^\sigma = \mathbf{P}^\tau$, respectively).

Let σ be a valuation with universe U and let $u \in U$. We define $\sigma(\mathbf{x}/u)$

$$\sigma(x/u)(s) = \begin{cases} \sigma(y) \text{ if } s \neq x \\ u \text{ if } s = x \end{cases}$$

Given an $\mathscr{L}$-valuation σ with universe U, we now define, for each $\mathscr{L}$-term $\mathbf{t}$, *the value of* $\mathbf{t}$ *under* σ (briefly, $\mathbf{t}^\sigma$) in such a way that $\mathbf{t}^\sigma \in U$.

Also for each $\mathscr{L}$-formula $\boldsymbol{\alpha}$ we define *the value of* $\boldsymbol{\alpha}$ *under* σ (briefly, $\boldsymbol{\alpha}^\sigma$) so that $\boldsymbol{\alpha}^\sigma$ is either $\top$ or $\bot$. This is done by recursion on deg $\mathbf{t}$ and deg $\boldsymbol{\alpha}$

(T1). If $\mathbf{x}$ is a variable, then $\mathbf{x}^\sigma$ is already defined.

(T2). If $\mathbf{f}$ is an n-ary function symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_n$ are $\mathscr{L}$-terms, then

$$(\mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_n)^\sigma = \mathbf{f}^\sigma(\mathbf{t}_1^\sigma,\ldots,\mathbf{t}_n^\sigma).$$

(F1). If $\mathbf{P}$ is an n-ary extralogical predicate symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_n$ are $\mathscr{L}$-terms, then

$$(\mathbf{P}\mathbf{t}_1\ldots\mathbf{t}_n)^\sigma = \begin{cases} \top & \text{if } \langle \mathbf{t}_1^\sigma,\ldots,\mathbf{t}_n^\sigma\rangle \in \mathbf{P}^\sigma, \\ \bot & \text{otherwise.} \end{cases}$$

(F1$^=$). If $\mathbf{s}$ and $\mathbf{t}$ are $\mathscr{L}$-terms and $\mathscr{L}$ is a language with equality, then

$$(\mathbf{s}=\mathbf{t})^\sigma = \begin{cases} \top & \text{if } \mathbf{s}^\sigma = \mathbf{t}^\sigma, \\ \bot & \text{otherwise.} \end{cases}$$

(F2). For every $\mathscr{L}$-formula $\boldsymbol{\beta}$,

$$(\neg\boldsymbol{\beta})^\sigma = \begin{cases} \top & \text{if } \boldsymbol{\beta}^\sigma = \bot, \\ \bot & \text{otherwise.} \end{cases}$$

(F3). For every $\mathscr{L}$-formula $\boldsymbol{\beta}$ and $\mathscr{L}$-formula $\boldsymbol{\gamma}$,

$$(\boldsymbol{\beta}\rightarrow\boldsymbol{\gamma})^\sigma = \begin{cases} \top & \text{if } \boldsymbol{\beta}^\sigma = \bot \text{ or } \boldsymbol{\gamma}^\sigma = \top, \\ \bot & \text{otherwise.} \end{cases}$$

(F4). For every $\mathscr{L}$-formula $\boldsymbol{\beta}$ and variable $\mathbf{x}$,

$$(\forall\mathbf{x}\boldsymbol{\beta})^\sigma = \begin{cases} \top & \text{if } \boldsymbol{\beta}^{\sigma(\mathbf{x}/u)} = \top \text{ for every } u \in U, \\ \bot & \text{otherwise,} \end{cases}$$

where U is the universe of σ.

The above definition will be referred to as "BSD". It must be stressed that what the BSD defines is not the valuation σ itself — which must be given *in advance,* by specifying a structure $\mathfrak{U}$ and an assignment of value $\mathbf{x}^\sigma \in U$ to each variable $\mathbf{x}$ — but two mappings *induced* by σ.

1.2. REMARK. Because of clause (F4), the BSD is strongly non-constructive: if U is infinite, (F4) does not provide us with a method for computing the value $(\forall \mathbf{x}\boldsymbol{\beta})^\sigma$ in a finite number of steps, for it presupposes the values $\boldsymbol{\beta}^{\sigma(\mathbf{x}/u)}$ for infinitely many u. This non-constructive character is inherited by all the semantic definitions given below, which are based on the BSD. Indeed, one of our main tasks will be to obtain a more constructive characterization of the concepts thus defined.

1.3. PROBLEM. Using Def. 1.5.1(g) show that $(\exists \mathbf{x}\boldsymbol{\alpha})^{\sigma} = \top$ iff $\boldsymbol{\alpha}^{\sigma(\mathbf{x}/u)} = \top$ for some $u \in U$, where U is the universe of the valuation σ.

1.4. PROBLEM. Show that $\exists$ could have been taken as primitive (i.e., as a symbol of $\mathscr{L}$) instead of $\forall$. (Replace BSD (F4) by the statement of Prob. 1.3 and replace Def. 1.5.1(g) by a definition of $\forall$ from which the original (F4) can be derived.)

We have noted above that, for every valuation σ, the induced mapping defined in clauses (F1)–(F4) of the BSD is a truth valuation. We shall say that σ *satisfies* a formula $\boldsymbol{\varphi}$ (or a set $\boldsymbol{\Phi}$ of formulas) — briefly, $\sigma \models \boldsymbol{\varphi}$ (or $\sigma \models \boldsymbol{\Phi}$, respectively) — if the truth valuation induced by σ satisfies $\boldsymbol{\varphi}$ (or $\boldsymbol{\Phi}$, respectively). Thus $\sigma \models \boldsymbol{\varphi}$ iff $\boldsymbol{\varphi}^\sigma = \top$; and $\sigma \models \boldsymbol{\Phi}$ iff $\boldsymbol{\varphi}^\sigma = \top$ for every $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$.

1.5. Definition. If every valuation satisfying a set $\boldsymbol{\Phi}$ of formulas also satisfies a formula $\boldsymbol{\alpha}$, we say that $\boldsymbol{\alpha}$ is a *logical consequence of* $\boldsymbol{\Phi}$ (or $\boldsymbol{\alpha}$ *follows logically from* $\boldsymbol{\Phi}$, or $\boldsymbol{\Phi}$ *logically entails* $\boldsymbol{\alpha}$). We write this briefly as "$\boldsymbol{\Phi} \models \boldsymbol{\alpha}$". As usual, we shall write "$\boldsymbol{\varphi} \models \boldsymbol{\alpha}$" instead of "$\{\boldsymbol{\varphi}\} \models \boldsymbol{\alpha}$" and say that $\boldsymbol{\alpha}$ is a logical consequence of $\boldsymbol{\varphi}$. If $\boldsymbol{\alpha}$ is satisfied by *every* valuation (i.e., if $\boldsymbol{\alpha}$ follows logically from the empty set of formulas), then we say that $\boldsymbol{\alpha}$ is *logically true* (or *logically valid*) and we write "$\models \boldsymbol{\alpha}$". If $\boldsymbol{\alpha} \models \boldsymbol{\beta}$ as well as $\boldsymbol{\beta} \models \boldsymbol{\alpha}$ (i.e., $\boldsymbol{\alpha}^\sigma = \boldsymbol{\beta}^\sigma$ for every valuation σ), we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *logically equivalent*. We say that a formula $\boldsymbol{\varphi}$ (or a set $\boldsymbol{\Phi}$ of formulas) is *satisfiable* if $\sigma \models \boldsymbol{\varphi}$ (or $\sigma \models \boldsymbol{\Phi}$, respectively) for some valuation σ.

1.7. PROBLEM. Show that $\Phi, \alpha \models \beta$ iff $\Phi \models \alpha \to \beta$. Hence show that $\{\varphi_1, \ldots, \varphi_n\} \models \beta$ iff $\models \varphi_1 \to \ldots \to \varphi_n \to \beta$. Also show that α and β are logically equivalent iff $\models \alpha \leftrightarrow \beta$.

check validity

$\alpha \to \beta \to \alpha,$

$(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma,$

$(\neg\alpha \to \beta) \to (\neg\alpha \to \neg\beta) \to \alpha,$

$(\alpha \to \beta) \wedge (\beta \to \gamma) \to \alpha \to \gamma,$

$(\alpha \to \beta) \wedge (\alpha \to \gamma) \to \alpha \to \beta \wedge \gamma,$

$(\alpha \to \gamma) \wedge (\beta \to \gamma) \to \alpha \vee \beta \to \gamma,$

$[\alpha \to (\beta \vee \gamma)] \to (\alpha \to \beta) \vee (\alpha \to \gamma).$

check equivalence

(a) $\alpha \to \beta$, $\neg\beta \to \neg\alpha$;

(b) $\neg(\alpha \to \beta)$, $\alpha \wedge \neg\beta$;

(c) $\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k)$, $\neg\varphi_1 \vee \neg\varphi_2 \vee \ldots \vee \neg\varphi_k$;

(d) $\neg(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_k)$, $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \ldots \wedge \neg\varphi_k$;

(e) $\alpha \wedge \beta \wedge \gamma$, $(\alpha \wedge \beta) \wedge \gamma$;

(f) $\alpha \vee \beta \vee \gamma$, $(\alpha \vee \beta) \vee \gamma$;

(g) $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k \to \alpha$, $\varphi_1 \to \varphi_2 \to \ldots \to \varphi_k \to \alpha$.

Freedom and bondage

2.1. THEOREM. *Let* $\mathbf{t}$ *be a term, and let* σ *and* τ *be valuations which agree on all variables and function symbols occurring in* $\mathbf{t}$. *Then* $\mathbf{t}^\sigma = \mathbf{t}^\tau$.

PROOF. By straightforward induction on $\deg \mathbf{t}$. If $\mathbf{t} = \mathbf{x}$ then, since $\mathbf{x}$ is a variable occurring in $\mathbf{t}$, we must have $\mathbf{x}^\sigma = \mathbf{x}^\tau$, i.e., $\mathbf{t}^\sigma = \mathbf{t}^\tau$.

If $\mathbf{t} = \mathbf{f}\mathbf{t}_1 \ldots \mathbf{t}_n$, where $\mathbf{f}$ is an n-ary function symbol and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are terms, then by assumption $\mathbf{f}^\sigma$ and $\mathbf{f}^\tau$ are the same. Also, since every variable or function symbol occurring in one of the arguments $\mathbf{t}_1, \ldots, \mathbf{t}_n$ occurs also in $\mathbf{t}$, we have by the induction hypothesis

$$\mathbf{t}_1^\sigma = \mathbf{t}_1^\tau, \ldots, \mathbf{t}_n^\sigma = \mathbf{t}_n^\tau.$$

Thus

$$\begin{aligned} \mathbf{t}^\sigma &= (\mathbf{f}\mathbf{t}_1 \ldots \mathbf{t}_n)^\sigma \\ &= \mathbf{f}^\sigma(\mathbf{t}_1^\sigma, \ldots, \mathbf{t}_n^\sigma) && \text{(by BSD (T2))} \\ &= \mathbf{f}^\tau(\mathbf{t}_1^\tau, \ldots, \mathbf{t}_n^\tau) && \text{(by ind. hyp.)} \\ &= (\mathbf{f}\mathbf{t}_1 \ldots \mathbf{t}_n)^\tau && \text{(by BSD (T2))} \\ &= \mathbf{t}^\tau. \end{aligned}$$

∎

We say that a term $\mathbf{t}$ is *closed* if it does not contain any variable.

a given occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *bound* if this occurrence is within a subformula of $\boldsymbol{\alpha}$ having the form $\forall \mathbf{x} \boldsymbol{\beta}$ (i.e., a universal subformula of $\boldsymbol{\alpha}$ which has $\mathbf{x}$ as variable of quantification); all other occurrences of $\mathbf{x}$ in $\boldsymbol{\alpha}$ are *free*.

2.2. DEFINITION. A given occurrence of a variable $\mathbf{x}$ in a formula $\boldsymbol{\alpha}$ is *free* in $\boldsymbol{\alpha}$ iff it is not *bound* in $\boldsymbol{\alpha}$. Moreover:

(1) If $\boldsymbol{\alpha}$ is atomic, then every occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *free* in $\boldsymbol{\alpha}$.

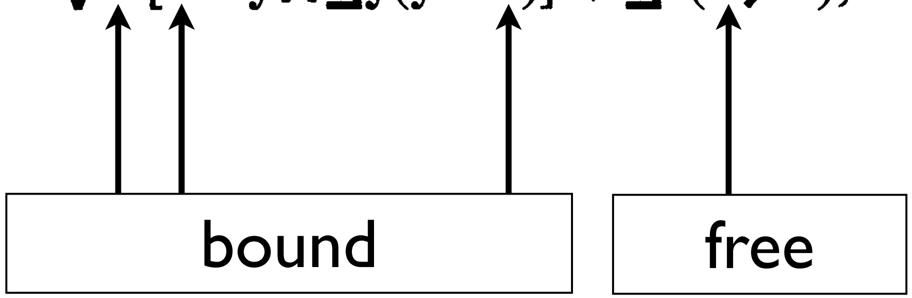
(2) If $\boldsymbol{\alpha} = \neg \boldsymbol{\beta}$, then a given occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *free* in $\boldsymbol{\alpha}$ iff the same occurrence is *free* in $\boldsymbol{\beta}$.

(3) If $\boldsymbol{\alpha} = \boldsymbol{\beta} \rightarrow \boldsymbol{\gamma}$, then a given occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *free* in $\boldsymbol{\alpha}$ iff that occurrence is a *free* occurrence of $\mathbf{x}$ in $\boldsymbol{\beta}$ or in $\boldsymbol{\gamma}$.

(4) If $\boldsymbol{\alpha} = \forall \mathbf{x} \boldsymbol{\beta}$, then every occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *bound* in $\boldsymbol{\alpha}$, but if $\boldsymbol{\alpha} = \forall \mathbf{y} \boldsymbol{\beta}$, where $\mathbf{y}$ is a variable other than $\mathbf{x}$, then a given occurrence of $\mathbf{x}$ in $\boldsymbol{\alpha}$ is *free* in $\boldsymbol{\alpha}$ iff that occurrence is *free* in $\boldsymbol{\beta}$.

Note that the same variable may have both free and bound occurrences in the same formula. For example, in the formula

$$\forall x\, [x = y \wedge \exists y (y = x)] \rightarrow \exists z (x \neq z),$$



definire induttivamente

FV(α)

We say that $\mathbf{x}$ is *free in* $\boldsymbol{\alpha}$ if $\mathbf{x}$ has at least one free occurrence in $\boldsymbol{\alpha}$.

The *free variables of* $\boldsymbol{\alpha}$ are the variables which are free in $\boldsymbol{\alpha}$.

dare definizione induttiva dell'insieme di variabili libere in una formula

2.3. THEOREM. *Let σ and τ be valuations which have the same universe U and which agree on every free variable of $\boldsymbol{\alpha}$ as well as on every extra-logical symbol occurring in $\boldsymbol{\alpha}$. Then $\boldsymbol{\alpha}^\sigma = \boldsymbol{\alpha}^\tau$.*

PROOF. By induction on $\deg \boldsymbol{\alpha}$. We deal here only with the case where $\boldsymbol{\alpha}$ is universal, leaving the other (and easier) cases to the reader.

Let $\boldsymbol{\alpha} = \forall \mathbf{x}\boldsymbol{\beta}$. Then, by the BSD, $\boldsymbol{\alpha}^\sigma = \top$ iff $\boldsymbol{\beta}^{\sigma(\mathbf{x}/u)} = \top$ for every u in the universe U of σ. Now, the extralogical symbols of $\boldsymbol{\beta}$ are exactly those of $\boldsymbol{\alpha}$. Also, the free variables of $\boldsymbol{\beta}$ are either exactly those of $\boldsymbol{\alpha}$, or they are those plus $\mathbf{x}$. But (for every $u \in U$) $\sigma(\mathbf{x}/u)$ and $\tau(\mathbf{x}/u)$ clearly agree not only on the free variables and extralogical symbols of $\boldsymbol{\alpha}$, but also on $\mathbf{x}$. Since $\deg \boldsymbol{\beta} < \deg \boldsymbol{\alpha}$, it follows from the induction hypothesis that $\boldsymbol{\beta}^{\sigma(\mathbf{x}/u)} = \boldsymbol{\beta}^{\tau(\mathbf{x}/u)}$. Thus $\boldsymbol{\alpha}^\sigma = \top$ iff $\boldsymbol{\beta}^{\tau(\mathbf{x}/u)} = \top$ for all $u \in U$, i.e., iff $\boldsymbol{\alpha}^\tau = \top$. ∎

2.4. PROBLEM. Show that if $\mathbf{x}$ is not free in $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$, $\forall \mathbf{x}\boldsymbol{\alpha}$ and $\exists \mathbf{x}\boldsymbol{\alpha}$ are logically equivalent.

A formula which has no free variables (so that all occurrences of variables in it, if any, are bound) is called a *sentence*. It follows from Thm. 2.3 that if $\boldsymbol{\alpha}$ is a sentence then the value $\boldsymbol{\alpha}^{\sigma}$ depends only on the structure $\mathfrak{U}$ underlying σ. In this case we define $\boldsymbol{\alpha}^{\mathfrak{U}}$ to be that value (i.e., $\boldsymbol{\alpha}^{\mathfrak{U}} = \boldsymbol{\alpha}^{\sigma}$ for any valuation σ which $\mathfrak{U}$ underlies).

If $\boldsymbol{\alpha}^{\mathfrak{U}} = \top$, we say that the structure $\mathfrak{U}$ *satisfies* the sentence $\boldsymbol{\alpha}$ (or $\boldsymbol{\alpha}$ *holds* in $\mathfrak{U}$, or $\mathfrak{U}$ is a *model* for $\boldsymbol{\alpha}$), briefly, $\mathfrak{U} \models \boldsymbol{\alpha}$. If $\mathfrak{U} \models \boldsymbol{\varphi}$ for every $\boldsymbol{\varphi}$ in a set $\boldsymbol{\Phi}$ of sentences, we say that $\mathfrak{U}$ is a *model* for $\boldsymbol{\Phi}$.

More generally, let $\boldsymbol{\alpha}$ be a formula such that all the free variables of $\boldsymbol{\alpha}$ are among the first k variables of $\mathscr{L}$, namely $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then, by Thm 2.3, $\boldsymbol{\alpha}^{\sigma}$ depends only on the structure $\mathfrak{U}$ underlying σ and on $\mathbf{v}_i^{\sigma}$ for $i = 1, \ldots, k$. We write

$$\mathfrak{U} \models \boldsymbol{\alpha}\,[u_1, \ldots, u_k]$$

when we wish to assert that $\sigma \models \boldsymbol{\alpha}$ for some (hence for every) valuation σ such that $\mathfrak{U}$ underlies σ and such that $\mathbf{v}_i^{\sigma} = u_i$ for $i = 1, \ldots, k$.

2.5. PROBLEM. Construct a sentence α containing only logical symbols (i.e., no function symbol and no predicate symbol other than $=$) such that α holds in a structure $\mathfrak{U}$ iff U has

(a) at least three members,
(b) at most three members,
(c) exactly three members.

2.6. PROBLEM. Using just one binary predicate symbol (but no other predicate symbols and no function symbols) construct a sentence α such that α has no finite model (i.e., no model with finite universe); but if U is any infinite set then α has a model whose universe is U.

2.7. REMARK. From Thm. 2.3 it follows that the various semantic concepts defined in Def. 1.5 are invariant with respect to language. For, if $\mathscr{L}$ and $\mathscr{L}'$ are two first-order languages and σ is an $\mathscr{L}$-valuation then there is an $\mathscr{L}'$-valuation σ' which agrees with σ on the symbols which $\mathscr{L}$ and $\mathscr{L}'$ have in common. Any formula α belonging to both $\mathscr{L}$ and $\mathscr{L}'$ will then get the same value under σ and σ'. Thus, e.g., if α is satisfiable as an $\mathscr{L}$-formula (i.e., satisfied by some $\mathscr{L}$-valuation) it is also satisfiable as an $\mathscr{L}'$-formula.

Substitution

Let s and **t** be terms. We define $\mathbf{s(x/t)}$ as the term obtained from **s** when an occurrence of **t** is substituted for each occurrence of **x** in **s**. In detail, $\mathbf{s(x/t)}$ is defined by recursion on deg **s** as follows:

3.1. DEFINITION. If $\mathbf{s}=\mathbf{x}$ then $\mathbf{s(x/t)}=\mathbf{t}$; but if $\mathbf{s}=\mathbf{y}$, where **y** is a variable other than **x**, then $\mathbf{s(x/t)}=\mathbf{y}$. If $\mathbf{s}=\mathbf{fs}_1\ldots\mathbf{s}_n$, where **f** is an n-ary function symbol and $\mathbf{s}_1,\ldots,\mathbf{s}_n$ are terms, then $\mathbf{s(x/t)}=\mathbf{fs}_1(\mathbf{x/t})\ldots\mathbf{s}_n(\mathbf{x/t})$.

3.2. THEOREM. *If* **s** *and* **t** *are terms,* **x** *a variable and* σ *a valuation, then*

$$\mathbf{s(x/t)}^{\sigma}=\mathbf{s}^{\sigma(\mathbf{x}/t)},$$

where $t=\mathbf{t}^{\sigma}$.

formulas

We shall first define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ only in those cases where the substitution of **t** for **x** in $\boldsymbol{\alpha}$ does not lead to "capture" and thus does not require any change of the variable of quantification. Later we shall also define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ in the remaining cases, by prescribing the changes that must be made in $\boldsymbol{\alpha}$ before the substitution may take place.

We shall say that **t** is *free to be substituted for* **x** *in* $\boldsymbol{\alpha}$ (briefly, *free for* **x** *in* $\boldsymbol{\alpha}$) if no free occurrence of **x** in $\boldsymbol{\alpha}$ is within a subformula of $\boldsymbol{\alpha}$ having the form $\forall \mathbf{y}\boldsymbol{\beta}$, where **y** occurs in **t**.

If **t** is free for **x** in $\boldsymbol{\alpha}$, we shall define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ as the result of substituting an occurrence of **t** for each free occurrence of **x** in $\boldsymbol{\alpha}$. (Note that because **t** is assumed to be free for **x** in $\boldsymbol{\alpha}$, all occurrences of variables that have been introduced *via* the substitution are free in $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$.)

3.3. DEFINITION. If $\boldsymbol{\alpha}$ is an atomic formula $\mathbf{Ps}_1\ldots\mathbf{s}_n$, then $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\alpha}$. And $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ is defined as $\mathbf{Ps}_1(\mathbf{x}/\mathbf{t})\ldots\mathbf{s}_n(\mathbf{x}/\mathbf{t})$. (Here, for $n=2$, $\mathbf{P}$ may also be the logical predicate symbol $=$.)

If $\boldsymbol{\alpha}=\neg\boldsymbol{\beta}$, then $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\alpha}$ iff $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\beta}$; if this is the case, $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ is defined to be $\neg[\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})]$.

If $\boldsymbol{\alpha}=\boldsymbol{\beta}\rightarrow\boldsymbol{\gamma}$, then $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\alpha}$ iff $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* both $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$; if this is the case we define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ as $\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})\rightarrow\boldsymbol{\gamma}(\mathbf{x}/\mathbf{t})$.

If $\boldsymbol{\alpha}=\forall\mathbf{y}\boldsymbol{\beta}$, then $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\alpha}$ iff one of the following conditions holds:

(a) $\mathbf{x}$ is not free in $\boldsymbol{\alpha}$,

(b) $\mathbf{x}$ is free in $\boldsymbol{\alpha}$ (hence, in particular, $\mathbf{x}\neq\mathbf{y}$), and $\mathbf{t}$ is *free for* $\mathbf{x}$ *in* $\boldsymbol{\beta}$, and $\mathbf{y}$ does not occur in $\mathbf{t}$.

In case (a) we define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ to be $\boldsymbol{\alpha}$. In case (b) we define $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$ to be $\forall\mathbf{y}\,[\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})]$.

It is easy to verify that if no variable occurring in $\mathbf{t}$ has a bound occurrence in $\boldsymbol{\alpha}$, then $\mathbf{t}$ is free for $\mathbf{x}$ in $\boldsymbol{\alpha}$. Also, $\mathbf{x}$ is always free for itself in $\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{x})=\boldsymbol{\alpha}$.

3.4. Theorem. *If* $\mathbf{t}$ *is free for* $\mathbf{x}$ *in* $\boldsymbol{\alpha}$ *then, for every valuation* σ,

$$\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})^{\sigma} = \boldsymbol{\alpha}^{\sigma(\mathbf{x}/t)},$$

where $t = \mathbf{t}^{\sigma}$.

Proof. By induction on $\deg \boldsymbol{\alpha}$. We distinguish various cases, corresponding to the cases in Def. 3.2. Here we only deal with the case $\boldsymbol{\alpha} = \forall \mathbf{y} \boldsymbol{\beta}$, leaving the other (easier) cases to the reader.

First suppose that $\mathbf{x}$ is not free in $\boldsymbol{\alpha}$. Then $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t}) = \boldsymbol{\alpha}$. Also, by Thm. 2.3, $\boldsymbol{\alpha}^{\sigma} = \boldsymbol{\alpha}^{\sigma(\mathbf{x}/t)}$. Thus

$$\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})^{\sigma} = \boldsymbol{\alpha}^{\sigma} = \boldsymbol{\alpha}^{\sigma(\mathbf{x}/t)}.$$

Now suppose that $\mathbf{x}$ is free in $\boldsymbol{\alpha}$ and $\mathbf{t}$ is free for $\mathbf{x}$ in $\boldsymbol{\beta}$ and $\mathbf{y}$ does not occur in $\mathbf{t}$. Then we have

(1) $$\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})^{\sigma} = (\forall \mathbf{y}\, [\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})])^{\sigma}.$$

By the BSD,

(2) $$(\forall \mathbf{y}\, [\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})])^{\sigma} = \top \quad \text{iff} \quad \boldsymbol{\beta}(\mathbf{x}/\mathbf{t})^{\sigma(\mathbf{y}/u)} = \top \quad \text{for all} \quad u \in U,$$

where U is the universe of σ. Since $\deg \boldsymbol{\beta} < \deg \boldsymbol{\alpha}$, the induction hypothesis yields

(3) $$\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})^{\sigma(\mathbf{y}/u)} = \boldsymbol{\beta}^{\sigma(\mathbf{y}/u)(\mathbf{x}/t')},$$

where $t' = \mathbf{t}^{\sigma(\mathbf{y}/u)}$. But $\mathbf{y}$ does not occur in $\mathbf{t}$. Hence by Thm. 2.1

$$t' = \mathbf{t}^{\sigma(\mathbf{y}/u)} = \mathbf{t}^{\sigma} = t.$$

Also, $\mathbf{x}$ and $\mathbf{y}$ are different (otherwise $\mathbf{x}$ could not be free in $\boldsymbol{\alpha}$); hence

$$\sigma(\mathbf{y}/u)(\mathbf{x}/t)=\sigma(\mathbf{x}/t)(\mathbf{y}/u).$$

For, it makes no difference whether we *first* change the value of $\mathbf{x}$ from $\mathbf{x}^\sigma$ to t and *then* change the value of $\mathbf{y}$ to u, or *vice versa.* (It *would* make a difference if $\mathbf{x}$ were the same as $\mathbf{y}$!) Hence we can rewrite (3) as

(4) $$\boldsymbol{\beta}(\mathbf{x}/\mathbf{t})^{\sigma(\mathbf{y}/u)}=\boldsymbol{\beta}^{\sigma(\mathbf{x}/t)(\mathbf{y}/u)}.$$

Now, by the BSD,

$$\boldsymbol{\beta}^{\sigma(\mathbf{x}/t)(\mathbf{y}/u)}=\top \text{ for all } u\in U \quad \text{iff} \quad [\forall \mathbf{y}\boldsymbol{\beta}]^{\sigma(\mathbf{x}/t)}=\top.$$

Combining this with (1), (2) and (4) we get the required result. ∎

3.5. DEFINITION. If $\mathbf{z}$ is a variable which is not free in $\boldsymbol{\beta}$ but is free for $\mathbf{x}$ in $\boldsymbol{\beta}$, we say that $\forall \mathbf{z}\,[\boldsymbol{\beta}(\mathbf{x}/\mathbf{z})]$ arises from $\forall \mathbf{x}\boldsymbol{\beta}$ by *(correct) alphabetic change*. (Note that if $\mathbf{z}$ does not occur at all in $\boldsymbol{\beta}$, then $\mathbf{z}$ certainly satisfies both of the above conditions.)

3.6. THEOREM. *If* $\forall \mathbf{z}\,[\boldsymbol{\beta}(\mathbf{x}/\mathbf{z})]$ *arises from* $\forall \mathbf{x}\boldsymbol{\beta}$ *by alphabetic change, then these two formulas are logically equivalent.*

Consider a given formula $\boldsymbol{\alpha}$. Suppose $\boldsymbol{\alpha}$ has a universal subformula, say $\forall \mathbf{y}\boldsymbol{\beta}$. Let us replace one occurrence of $\forall \mathbf{y}\boldsymbol{\beta}$ in $\boldsymbol{\alpha}$ by an occurrence of a formula $\forall \mathbf{z}\,[\boldsymbol{\beta}(\mathbf{y}/\mathbf{z})]$ arising from $\forall \mathbf{y}\boldsymbol{\beta}$ by alphabetic change (i.e., $\mathbf{z}$ is not free in $\boldsymbol{\beta}$, but is free for $\mathbf{y}$ in $\boldsymbol{\beta}$). We shall say that $\boldsymbol{\alpha}'$ is a *variant* of $\boldsymbol{\alpha}$ (briefly, $\boldsymbol{\alpha} \sim \boldsymbol{\alpha}'$) if $\boldsymbol{\alpha}$ can be transformed into $\boldsymbol{\alpha}'$ by a finite number of applications of steps like the one just described. (We include the case where the number of such steps is 0, so that $\boldsymbol{\alpha} \sim \boldsymbol{\alpha}$.)

3.7. DEFINITION. If α is atomic, then α is its own sole *variant*.

If $\alpha = \neg\beta$, then the *variants* of α are all formulas of the form $\neg(\beta')$, where β' is a *variant* of β.

If $\alpha = \beta \to \gamma$, then the *variants* of α are all formulas of the form $\beta' \to \gamma'$, where β' and γ' are *variants* of β and γ respectively.

If $\alpha = \forall y \beta$, then the *variants* of α are all formulas $\forall y \beta'$, where β' is a *variant* of β, as well as all formulas $\forall z\, [\beta'(y/z)]$ obtained from such $\forall y \beta'$ by alphabetic change.

(i) Equivalence of formulas: in fact what we call a formula is indeed an equivalence class: we identify two formulas which differ only by the names of their bound variables, precisely: $A \sim A$; if $A \sim A'$ and $B \sim B'$, then $\neg A \sim \neg A'$, $A \wedge B \sim A' \wedge B'$, $A \vee B \sim A' \vee B'$, $A \to B \sim A' \to B'$. If $A\,[x_n]$ and $A'[x_m]$ are formulas, let x_p be a variable occurring neither in A nor in A'; then, if $A\,[x_p] \sim A'\,[x_p]$ we have $\forall x_n A\,[x_n] \sim \forall x_m A'\,[x_m]$ and $\exists x_n A\,[x_n] \sim \exists x_m A\,[x_m]$. An immediate consequence of the definition is that, given C, it is possible to find D such that $C \sim D$ and

– no variable in D is both free and bound

– any bound variable in D occurs in the scope of only one occurrence of a quantifier.