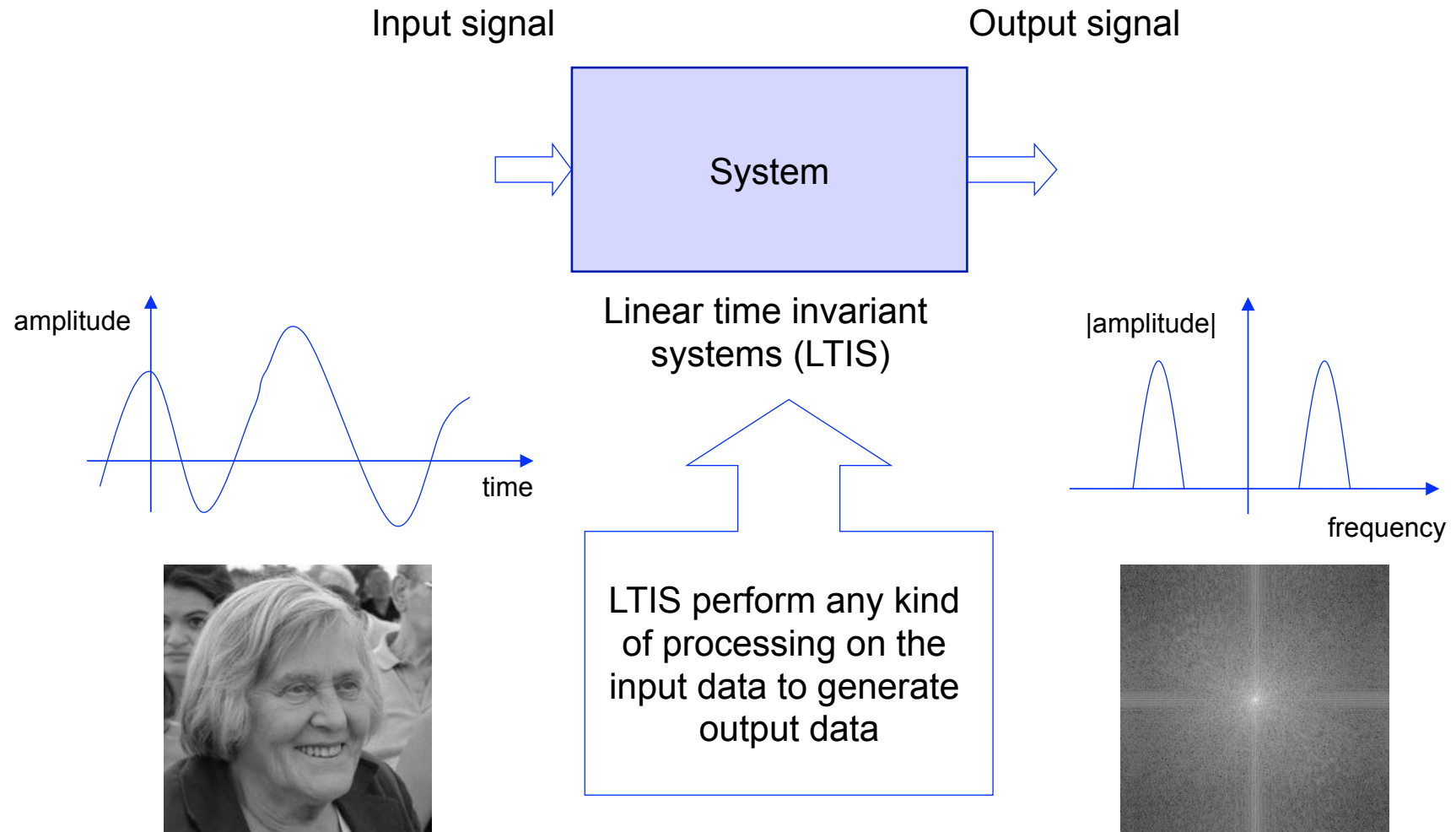# Basics of Signals and Systems

Gloria Menegaz

AA 2013-2014

Didactic material

- Textbook
    - Signal Processing and Linear Systems, B.P. Lathi, CRC Press
- Other books
    - Signals and Systems, Richard Baraniuk's lecture notes, available on line
    - Digital Signal Processing (4th Edition) (Hardcover), John G. Proakis, Dimitris K Manolakis
    - Teoria dei segnali analogici, M. Luise, G.M. Vitetta, A.A. D'Amico, McGraw-Hill
    - Signal processing and linear systems, Schaun's outline of digital signal processing
- All textbooks are available at the library
- Handwritten notes will be available on demand

Signals&Systems

Input signal

Output signal

System

Linear time invariant systems (LTIS)

amplitude

time

|amplitude|

frequency

LTIS perform any kind of processing on the input data to generate output data

Contents

Signals

- **Signal classification and representation**
  - Types of signals
  - Sampling theory
  - Quantization

- **Signal analysis**
  - Fourier Transform
    - Continuous time, Fourier series, Discrete Time Fourier Transforms, Windowed FT
  - Spectral Analysis

Systems

- **Linear Time-Invariant Systems**
  - Time and frequency domain analysis
  - Impulse response
  - Stability criteria

- **Digital filters**
  - Finite Impulse Response (FIR)

- **Mathematical tools**
  - Laplace Transform
    - Basics
 - Z-Transform
    - Basics
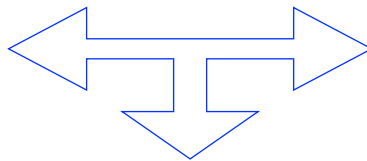
Applications in the domain of Bioinformatics

What is a signal?

- A signal is a set of information of data

  - Any kind of physical variable subject to variations represents a signal
  - Both the independent variable and the physical variable can be either scalars or vectors
    - Independent variable: time (t), space (x, $\mathbf{x}=[x_1,x_2]$, $\mathbf{x}=[x_1,x_2,x_3]$)
    - Signal:
    - Electrochardiography signal (EEG) 1D, voice 1D, music 1D
    - Images (2D), video sequences (2D+time), volumetric data (3D)
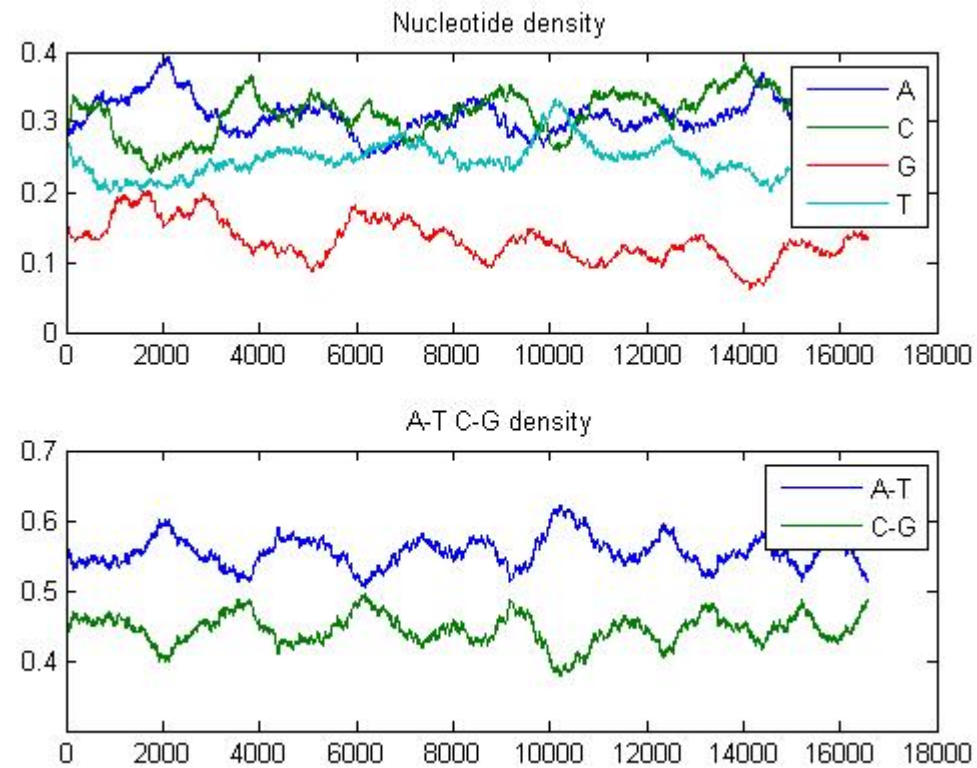
Example: 1D biological signals: ECG
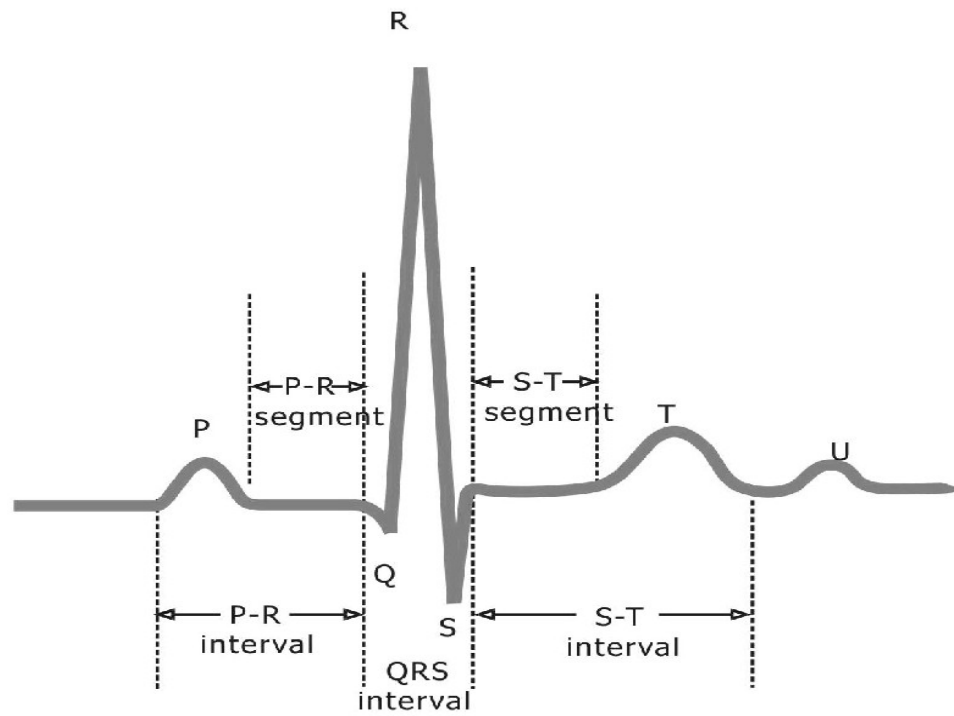
Example: 1D biological signals: EEG
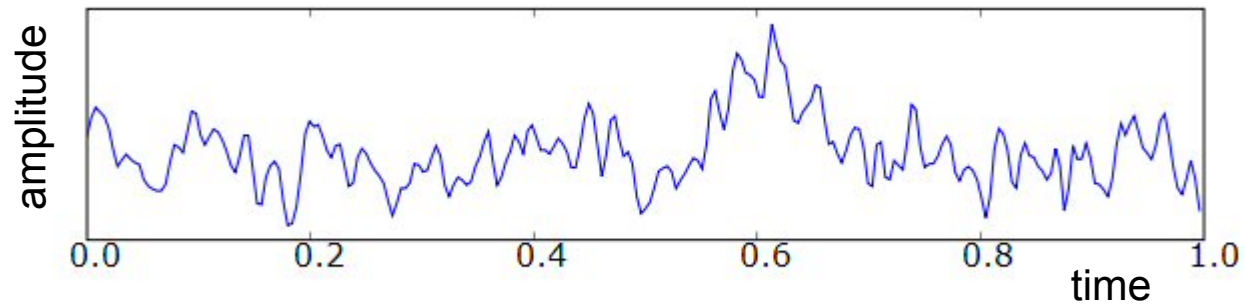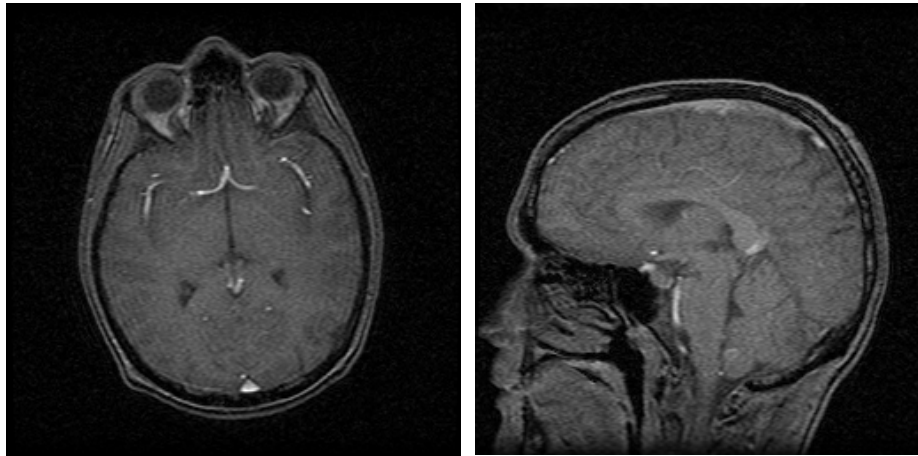
1D biological signals: DNA sequencing

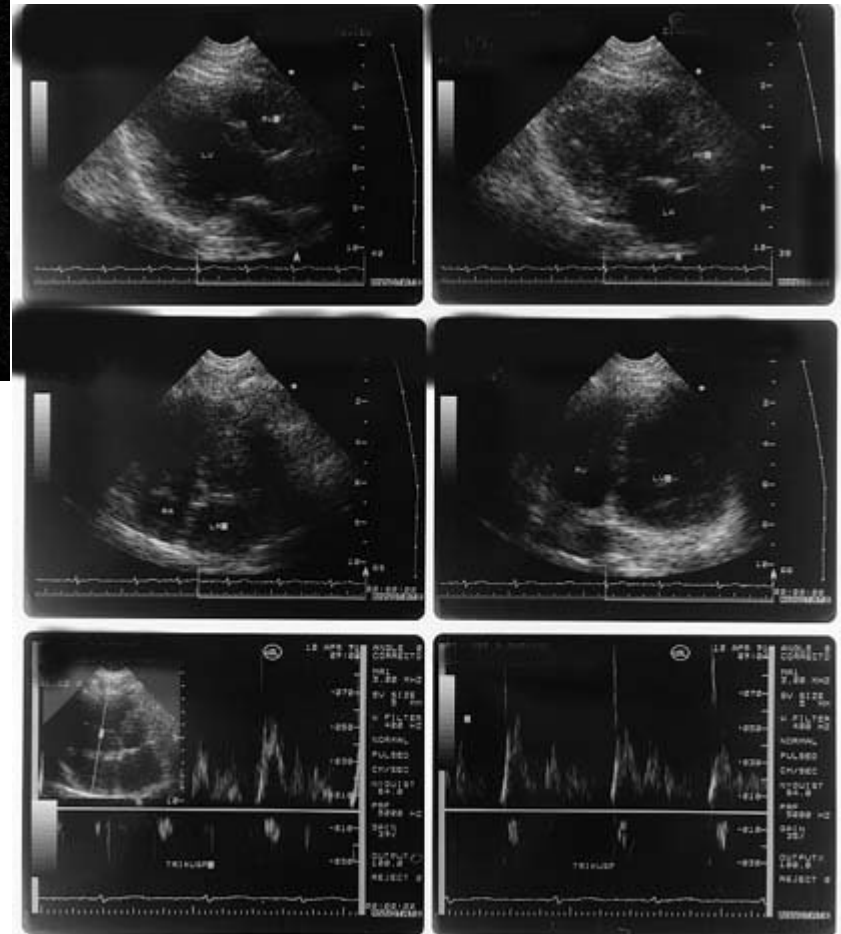GATCACAGGTCTATCACCCTATTAACCACTCACGGGAGCTCTCCATG......
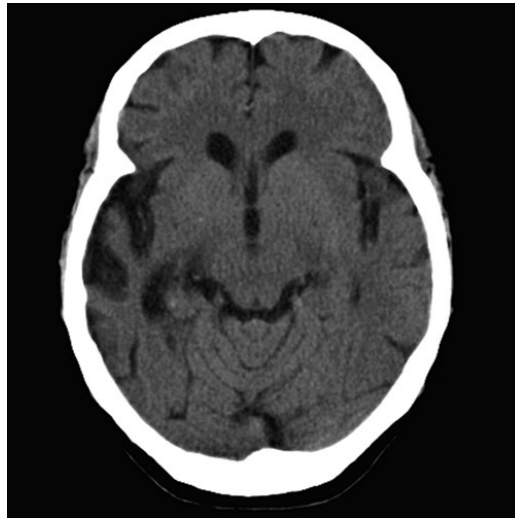
Example: 2D biological signals: MI
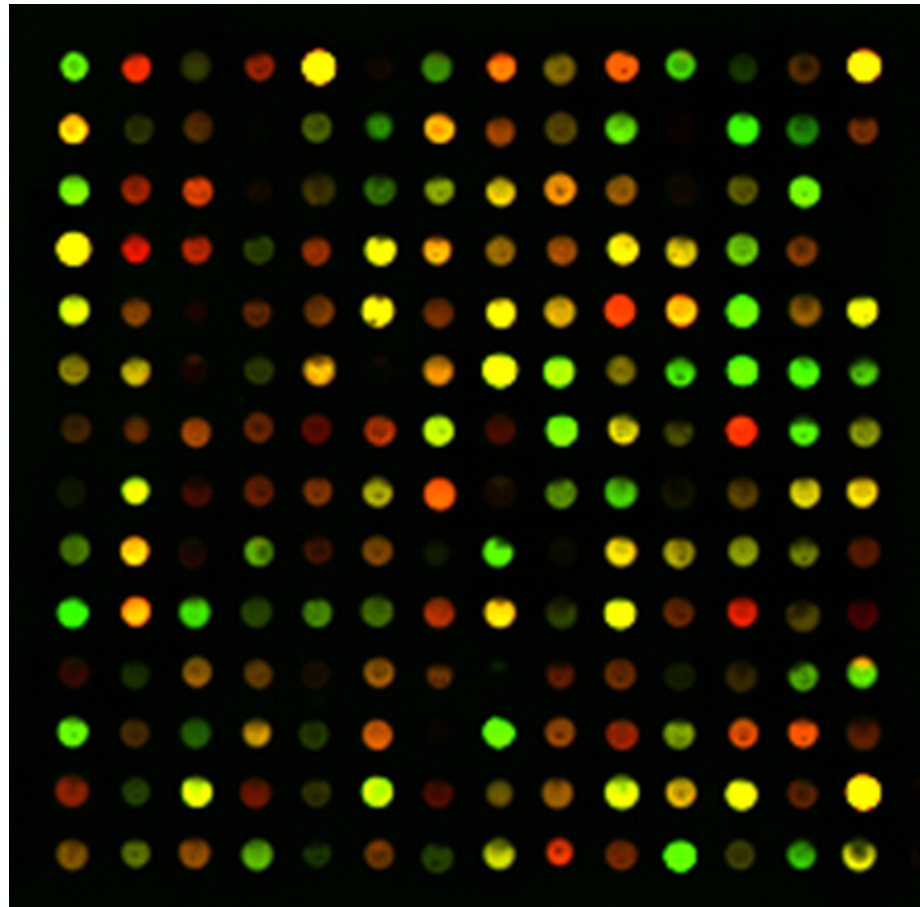
MRI

US

CT

Example: 2D biological signals: microarrays
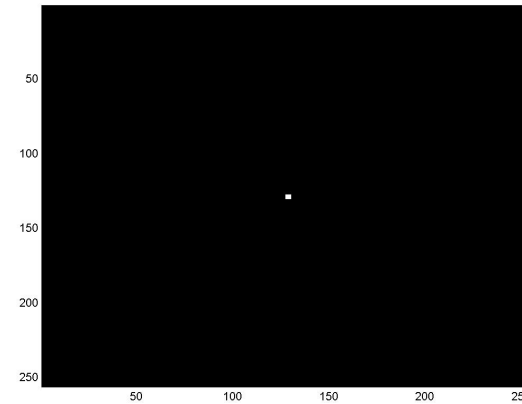
Signals as functions

- **Continuous functions of real independent variables**

  - 1D: $f=f(x)$
  - 2D: $f=f(x,y)$ x,y
  - Real world signals (audio, ECG, images)

- **Real valued functions of discrete variables**

  - 1D: $f=f[k]$
  - 2D: $f=f[i,j]$
  - *Sampled* signals

- **Discrete functions of discrete variables**

  - 1D: $f^d=f^d[k]$
  - 2D: $f^d=f^d[i,j]$
  - *Sampled and quantized* signals

Images as functions

- Gray scale images: 2D functions

  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations

  - Set of values taken by the function : gray levels

- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$

  - I,J: number of rows (columns) of the matrix corresponding to the image

  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i, j \neq 0; i \neq j \end{cases}$$

$$\delta[i, j - J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$
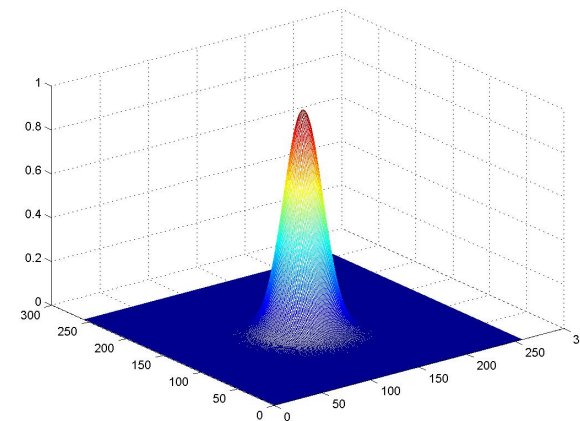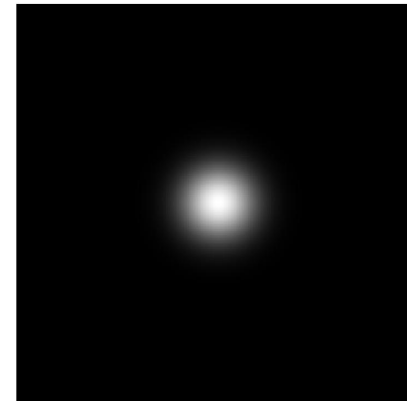

Gloria Menegaz
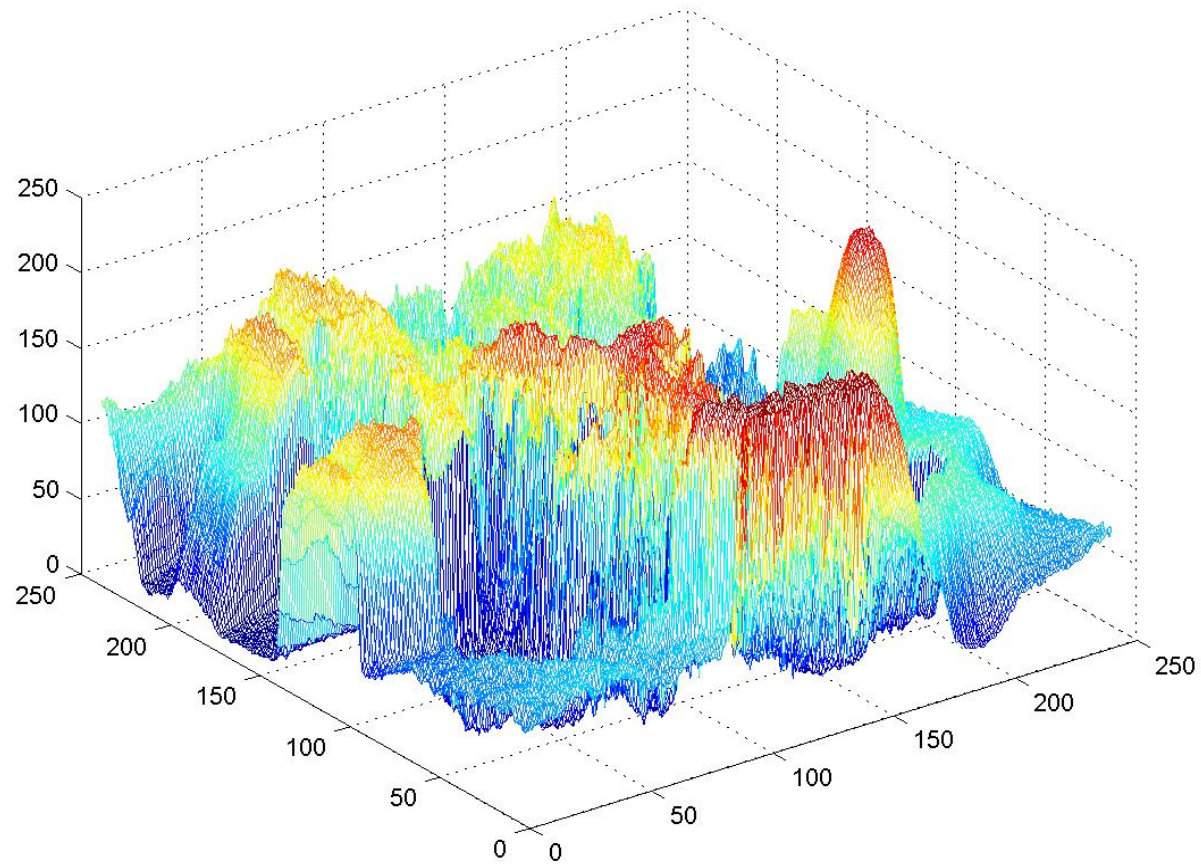
13
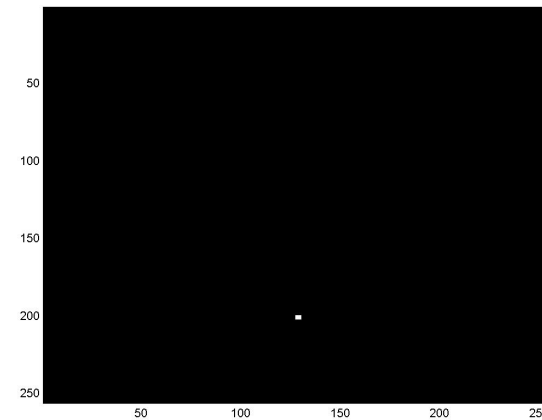
Example 2: Gaussian

Continuous function

$$f(x, y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{x^2 + y^2}{2\sigma^2}}$$

Discrete version

$$f[i, j] = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{i^2 + j^2}{2\sigma^2}}$$

Example 3: Natural image

Example 3: Natural image

What is a system?

- Systems process signals to
  - Extract information (DNA sequence analysis)
  - Enable transmission over channels with limited capacity (JPEG, JPEG2000, MPEG coding)
  - Improve security over networks (encryption, watermarking)
  - Support the formulation of diagnosis and treatment planning (medical imaging)
 - ....

closed-loop

input → System → output
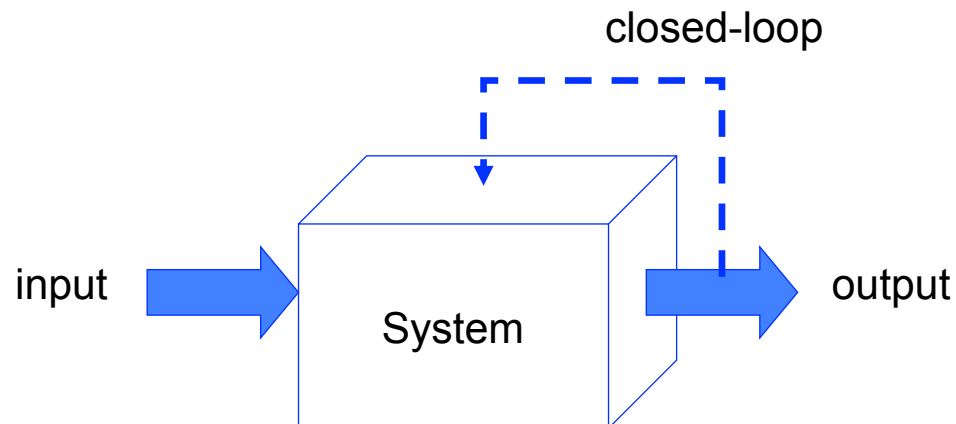
The function linking the output of the system with the input signal is called transfer function and it is typically indicated with the symbol $h(\cdot)$

Classification of signals

- Continuous time – Discrete time

- Analog – Digital (numerical)

- Periodic – Aperiodic

- Energy – Power

- Deterministic – Random (probabilistic)

- Note

  – Such classes are not disjoint, so there are digital signals that are periodic of power type and others that are aperiodic of power type etc.

  – Any combination of single features from the different classes is possible

Continuous time – discrete time

- Continuous time signal: a signal that is specified for every real value of the independent variable
    - The independent variable is continuous, that is it takes any value on the real axis
    - The domain of the function representing the signal has the cardinality of real numbers
        - Signal ↔ f=f(t)
        - Independent variable ↔ time (t), position (x)
        - For continuous-time signals: $t \in \mathbb{R}$
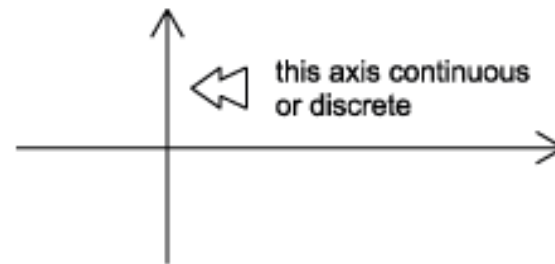
amplitude
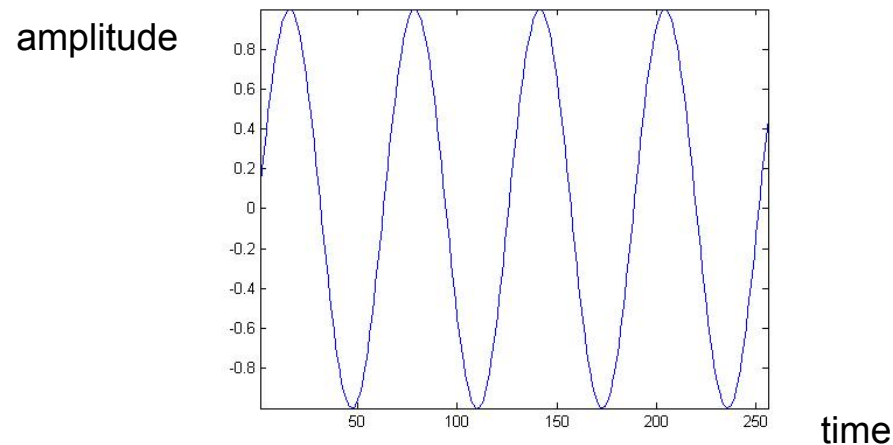
time

Continuous time – discrete time

- Discrete time signal: a signal that is specified only for *discrete values* of the independent variable

  - It is usually generated by *sampling* so it will only have values at *equally spaced* intervals along the time axis

  - The domain of the function representing the signal has the cardinality of integer numbers

    - Signal $\leftrightarrow$ f=f[n], also called "sequence"

    - Independent variable $\leftrightarrow$ n

    - For discrete-time functions: $t \in \mathbf{Z}$

amplitude

this axis continuous
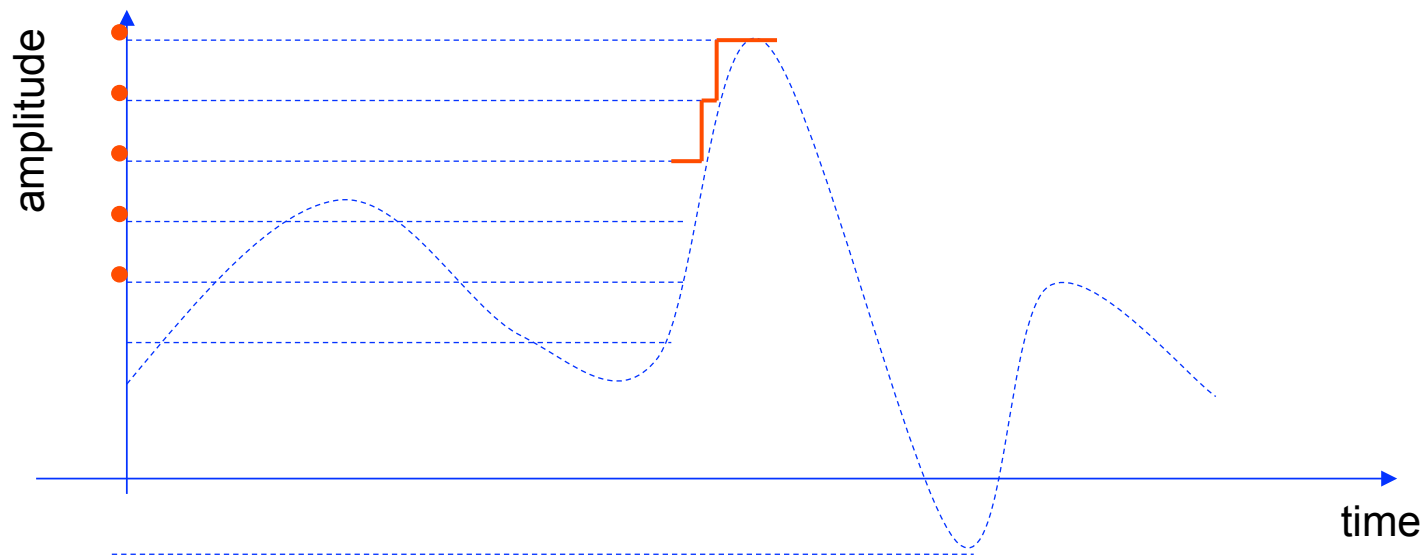or discrete

time (discrete)
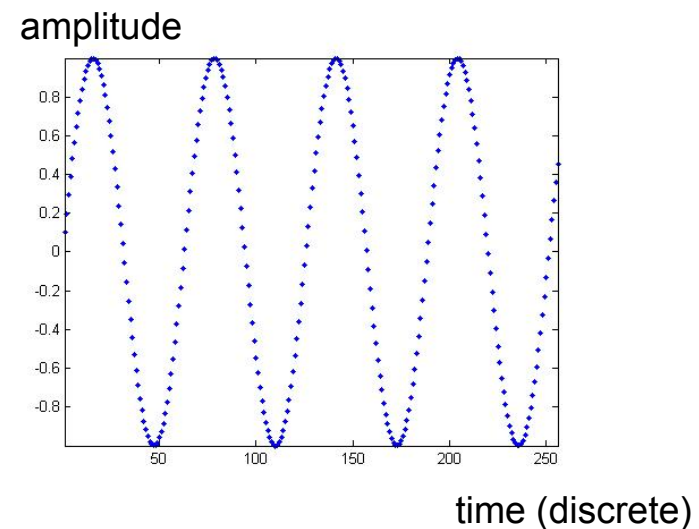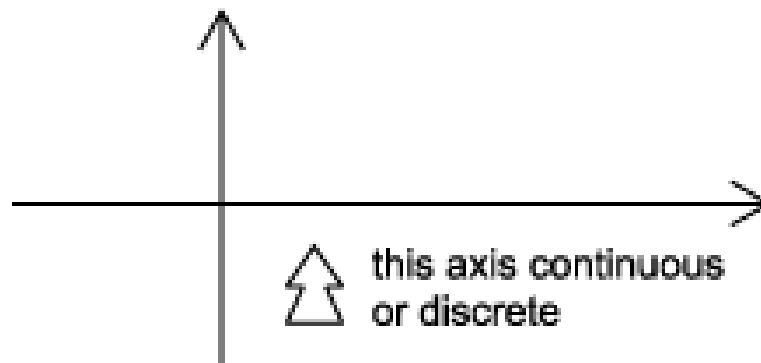
Analog - Digital

- **Analog signal**: signal whose amplitude can take on any value in a continuous range

    - The amplitude of the function f(t) (or f(x)) has the cardinality of real numbers
        - The difference between analog and digital is similar to the difference between continuous-time and discrete-time. In this case, however, the difference is with respect to the value of the function (y-axis)
    - Analog corresponds to a continuous y-axis, while digital corresponds to a discrete y-axis

this axis continuous or discrete

- *Here we call digital what we have called quantized in the EI class*

- An analog signal can be both continuous time and discrete time
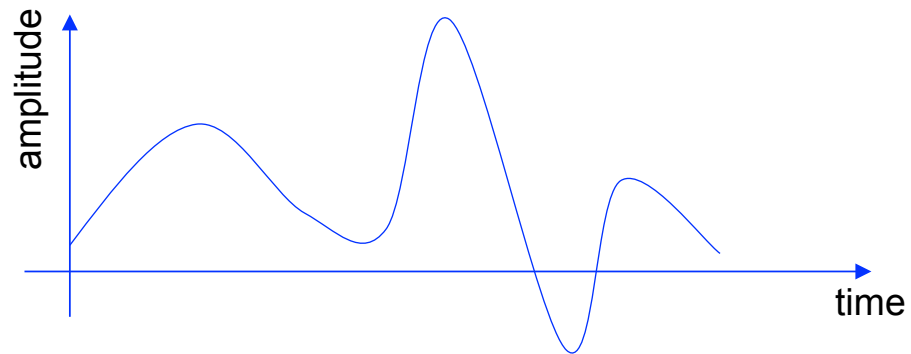
Analog - Digital

- **Digital signal**: a signal is one whose amplitude can take on only a finite number of values (thus it is quantized)
  - The amplitude of the function f() can take only a finite number of values
  - A digital signal whose amplitude can take only M different values is said to be M-ary
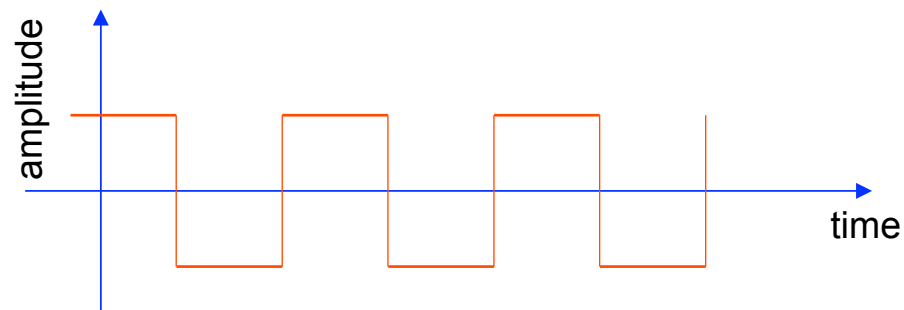   - Binary signals are a special case for M=2
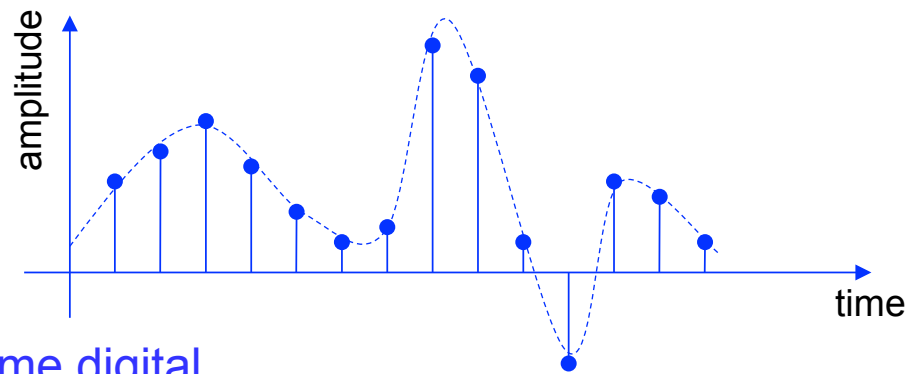
Example

– Continuous time analog

– Continuous time digital (or quantized)

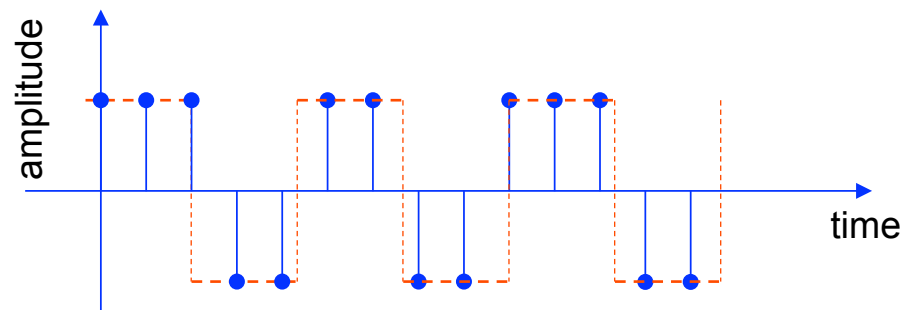- binary sequence, where the values of the function can only be one or zero.

Example

- Discrete time analog

- Discrete time digital

  - binary sequence, where the values of the function can only be one or zero.

Summary

Signal amplitude/ Time or space	Real	Integer
Real	Analog Continuous-time	Digital Continuous-time
Integer	Analog Discrete-time	Digital Discrete time

Note

- In the image processing class we have defined as digital those signals that are both quantized and discrete time. It is a more restricted definition.

- The definition used here is as in the Lathi book.

Periodic - Aperiodic

- A signal f(t) is *periodic* if there exists a positive constant T_0 such that

$$f(t + T_0) = f(t) \qquad \forall t$$

  - The *smallest* value of T_0 which satisfies such relation is said the *period* of the function f(t)
  - A periodic signal remains unchanged when *time-shifted* of integer multiples of the period
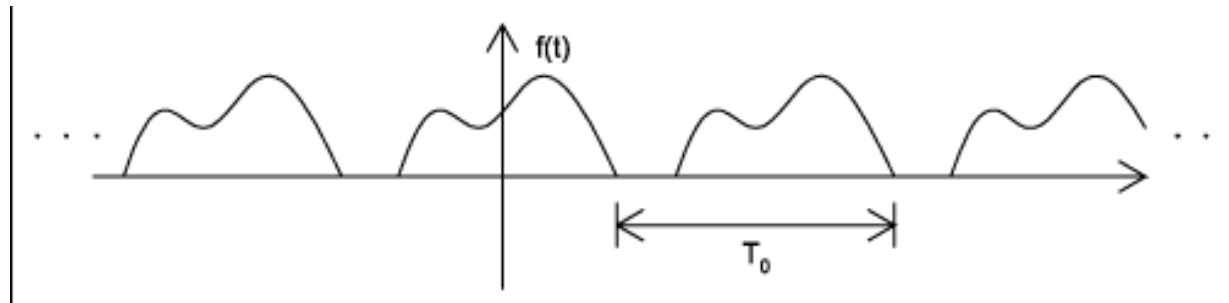 - Therefore, by definition, it starts at minus infinity and lasts forever

$$-\infty \le t \le +\infty \qquad t \in {}^\circ$$

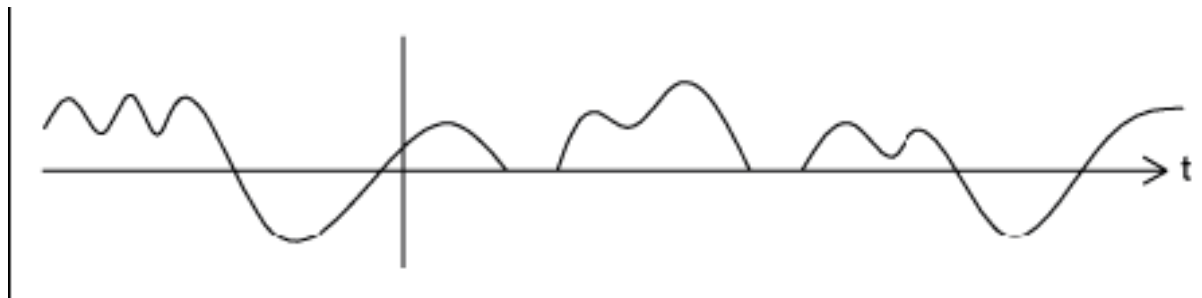$$-\infty \le n \le +\infty \qquad n \in \mathbf{Z}$$

  - Periodic signals can be generated by *periodical extension*
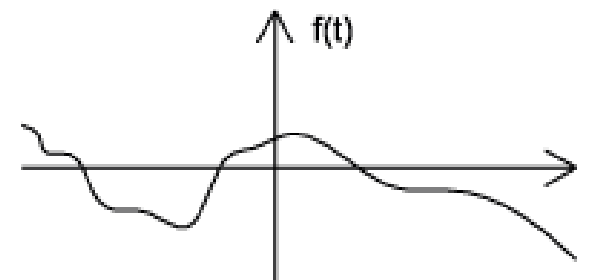
Examples

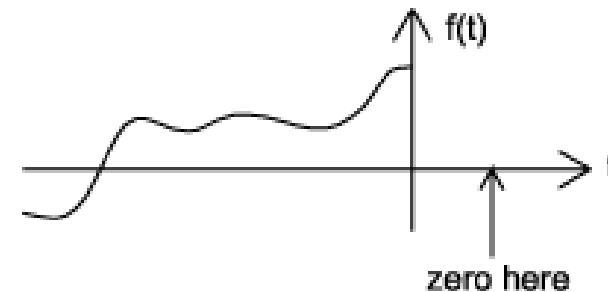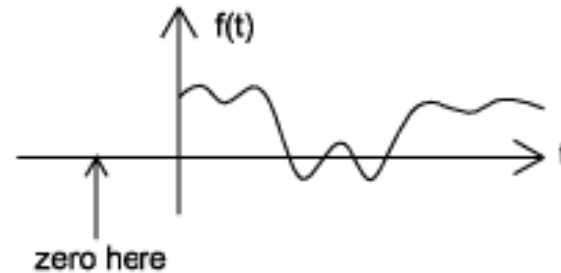- Periodic signal with period T_0

- Aperiodic signal

Causal and non-Causal signals

- *Causal* signals are signals that are *zero for all negative time (or spatial positions)*, while

- *Anticausal* are signals that are zero for all *positive time (or spatial positions)*.

- *Noncausal* signals are signals that have nonzero values in both positive and negative time

Causal and non-causal signals

- Causal signals

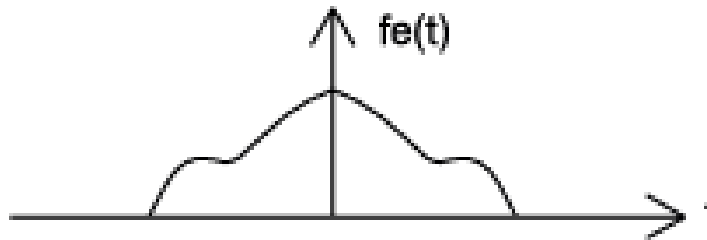$$f(t) = 0 \qquad t < 0$$

- Anticausals signals
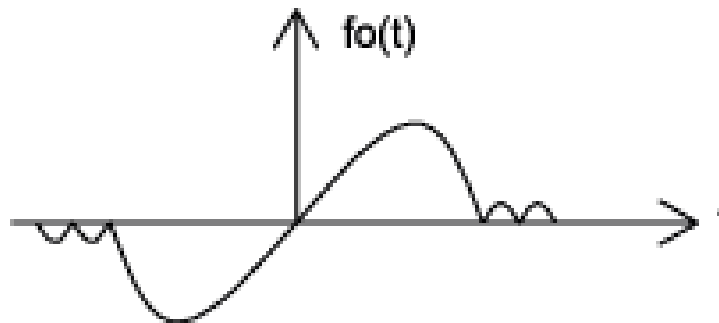
$$f(t) = 0 \qquad t \geq 0$$

- Non-causal signals

$$\exists t_1 < 0 : \qquad f(t_1) = 0$$

Even and Odd signals

- An even signal is any signal f such that f (t) = f (-t). Even signals can be easily spotted as they are symmetric around the vertical axis.

- An odd signal, on the other hand, is a signal f such that f (t)= - (f (-t))

Decomposition in even and odd components

- Any signal can be written as a combination of an even and an odd signals
  - Even and odd components

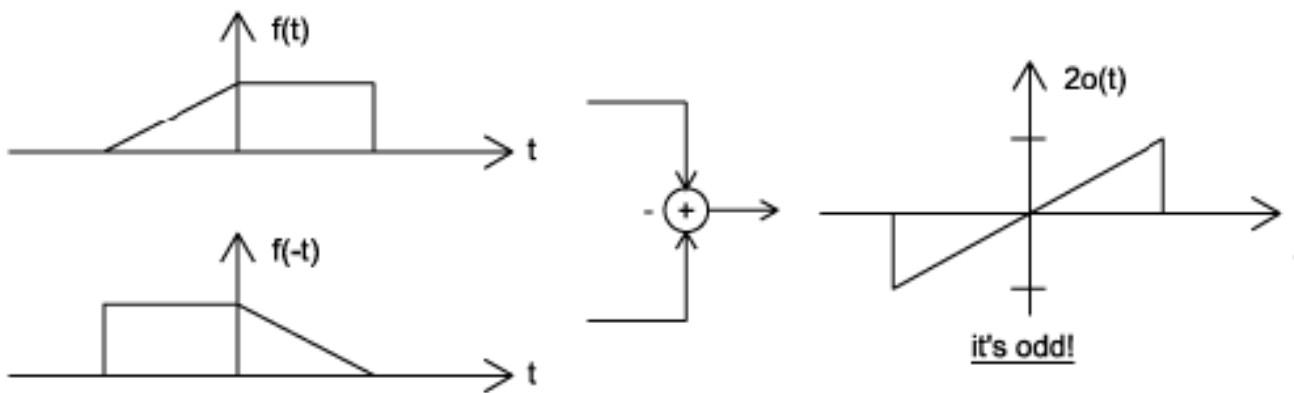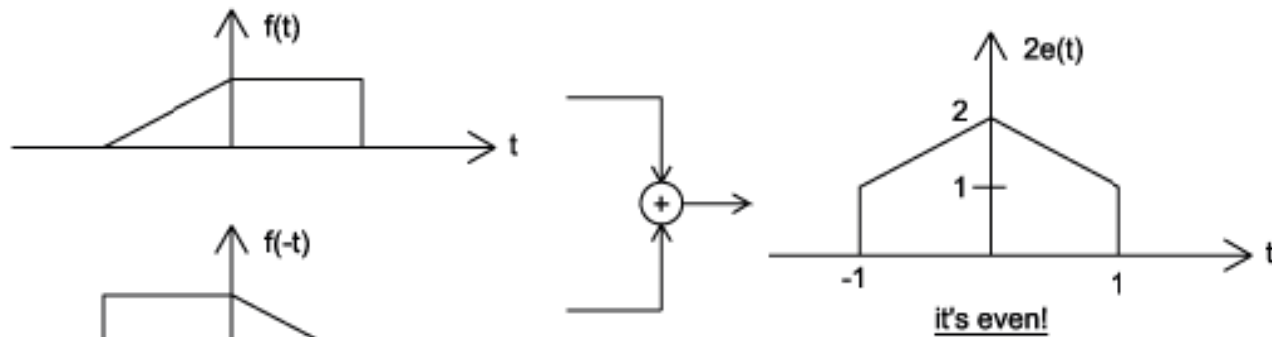$$f(t) = \frac{1}{2}\left(f(t) + f(-t)\right) + \frac{1}{2}\left(f(t) - f(-t)\right)$$
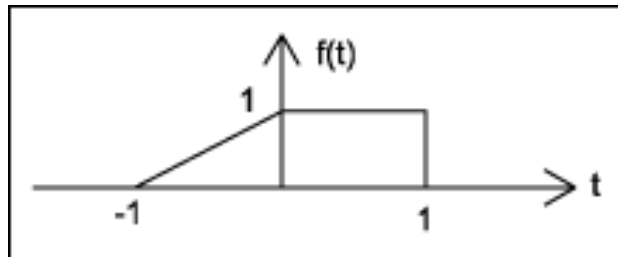
$$f_e(t) = \frac{1}{2}\left(f(t) + f(-t)\right) \quad \text{even component}$$

$$f_o(t) = \frac{1}{2}\left(f(t) - f(-t)\right) \quad \text{odd component}$$
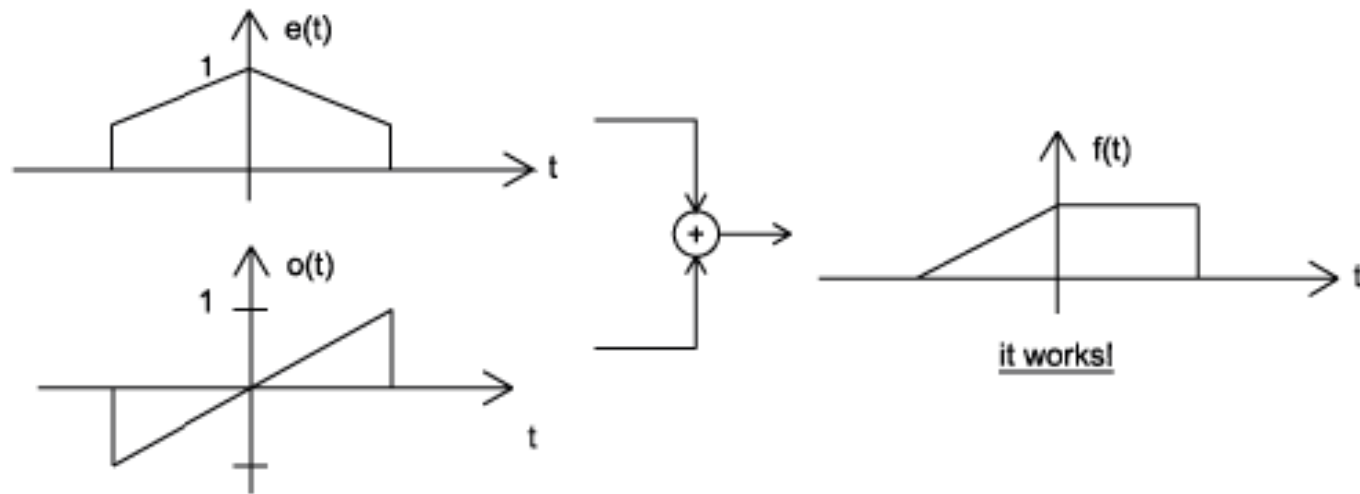
$$f(t) = f_e(t) + f_o(t)$$

Example

Example

- Proof

Some properties of even and odd functions

- even function x odd function = odd function

- odd function x odd function = even function

- even function x even function = even function

- Area

$$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t)\,dt$$

$$\int_{-a}^{a} f_o(t)\,dt = 0$$

Deterministic - Probabilistic

- Deterministic signal: a signal whose *physical description* in known completely

- A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table.

- Because of this the future values of the signal can be calculated from past values with complete confidence.

  – There is *no uncertainty* about its amplitude values

  – Examples: signals defined through a mathematical function or graph

- Probabilistic (or random) signals: the amplitude values *cannot be predicted precisely* but are known only in terms of probabilistic descriptors

- The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals

  – They are realization of a stochastic process for which a model could be available

  – Examples: EEG, evocated potentials, noise in CCD capture devices for digital cameras

Example

- Deterministic signal
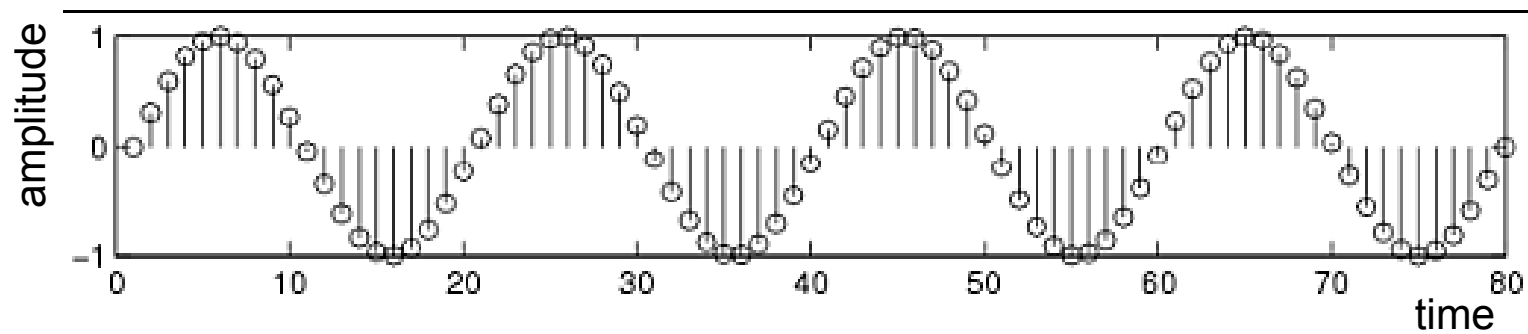
- Random signal
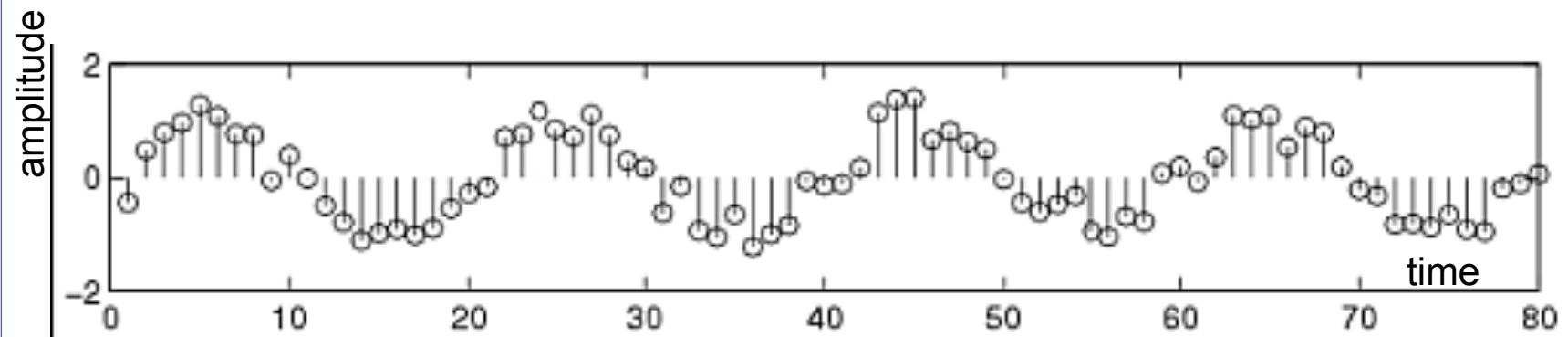
Finite and Infinite length signals

- A finite length signal is non-zero over a finite set of values of the independent variable

$$f = f(t), \forall t : t_1 \leq t \leq t_2$$

$$t_1 > -\infty, t_2 < +\infty$$

- An infinite length signal is non zero over an infinite set of values of the independent variable

  - For instance, a sinusoid f(t)=sin(ωt) is an infinite length signal

Size of a signal: Norms

- "Size" indicates largeness or strength.

- We will use the mathematical concept of the norm to quantify this notion for both continuous-time and discrete-time signals.

- The energy is represented by the area under the curve (of the squared signal)
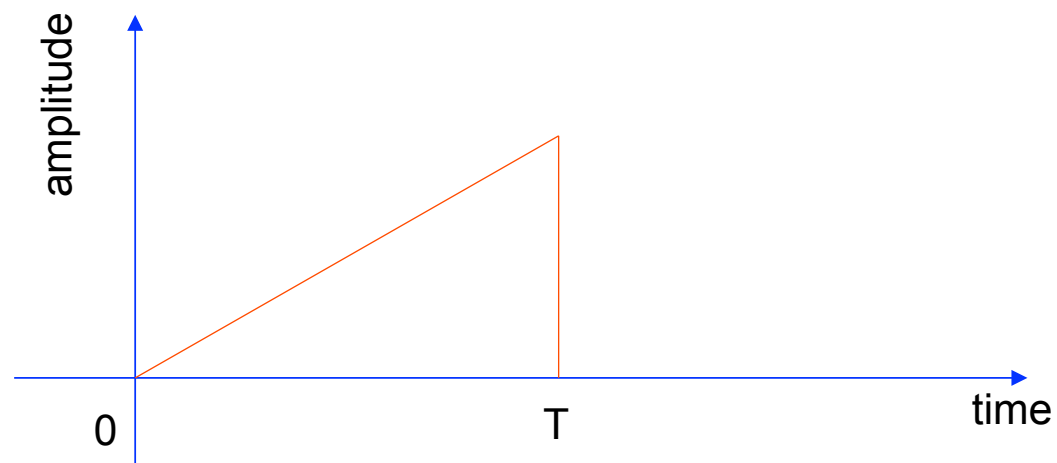
Energy

- Signal energy

$$E_f = \int_{-\infty}^{+\infty} f^2(t)\,dt$$

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2\,dt$$

- Generalized energy : L_p norm

  - For p=2 we get the energy (L_2 norm)

$$\|f(t)\| = \left(\int (|f(t)|)^p\,dt \right)^{1/p}$$

$$1 \le p < +\infty$$

Power

- Power
  - The power is the time average (mean) of the squared signal amplitude, that is the *mean-squared* value of f(t)

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^2(t)dt$$

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

Power - Energy

- The square root of the power is the root mean square (*rms*) value

    - This is a very important quantity as it is the most widespread measure of similarity/dissimilarity among signals

    - It is the basis for the definition of the Signal to Noise Ratio (SNR)

$$SNR = 20\log_{10}\left(\sqrt{\frac{P_{signal}}{P_{noise}}}\right)$$

    - It is such that a constant signal whose amplitude is =rms holds the same power content of the signal itself

- There exists signals for which neither the energy nor the power are finite
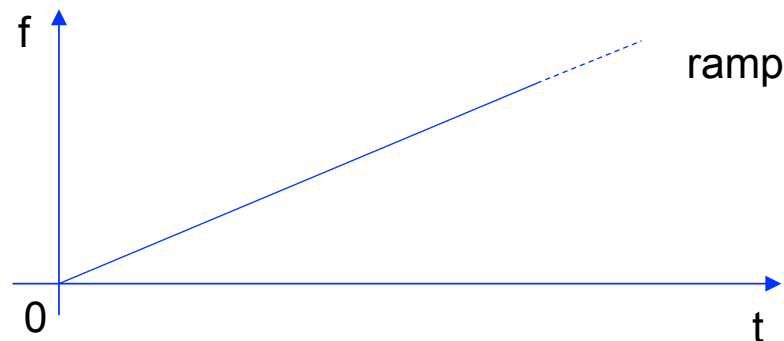
Energy and Power signals

- A signal with finite energy is an energy signal

    - Necessary condition for a signal to be of energy type is that the amplitude goes to zero as the independent variable tends to infinity

- A signal with finite and different from zero power is a power signal

    - The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity

    - A power signal has infinite energy and an energy signal has zero power

    - There exist signals that are neither power nor energy, such as the ramp

- All practical signals have finite energy and thus are energy signals

    - It is impossible to generate a real power signal because this would have infinite duration and infinite energy, which is not doable.
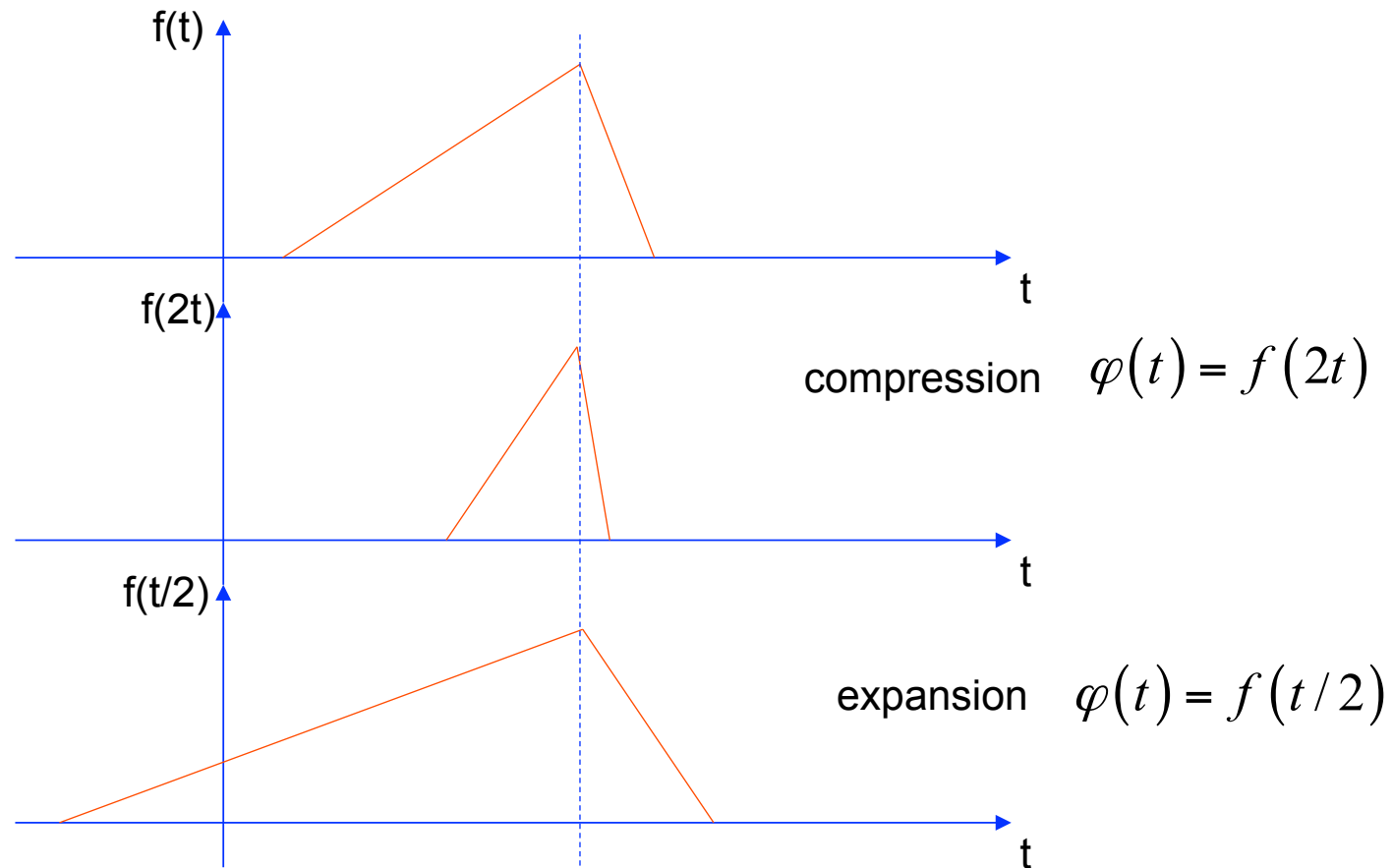
Useful signal operations: shifting, scaling, inversion

- **Shifting**: consider a signal f(t) and the same signal delayed/anticipated by T seconds

f(t)

T>0

f(t+T)

anticipated

T

f(t-T)

delayed

T
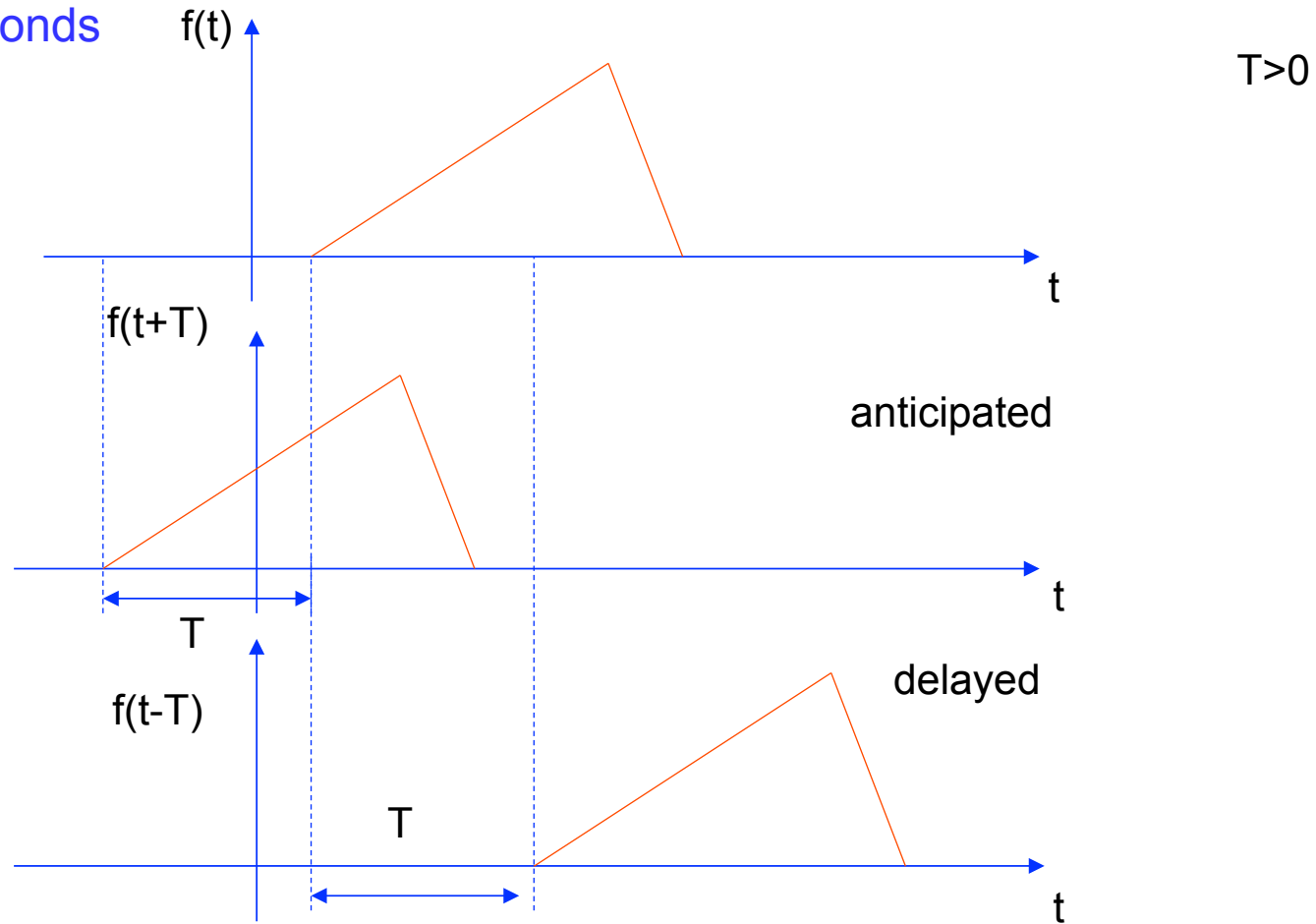
Useful signal operations: shifting, scaling, inversion

- (Time) Scaling: compression or expansion of a signal in time

f(t)

f(2t)

compression $\varphi(t) = f(2t)$

f(t/2)

expansion $\varphi(t) = f(t/2)$

Useful signal operations: shifting, scaling, inversion

- Scaling: generalization

$$a > 1$$

$$\varphi(t) = f(at) \rightarrow \text{ compressed version}$$

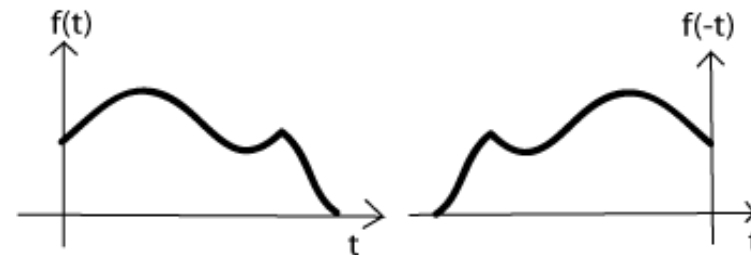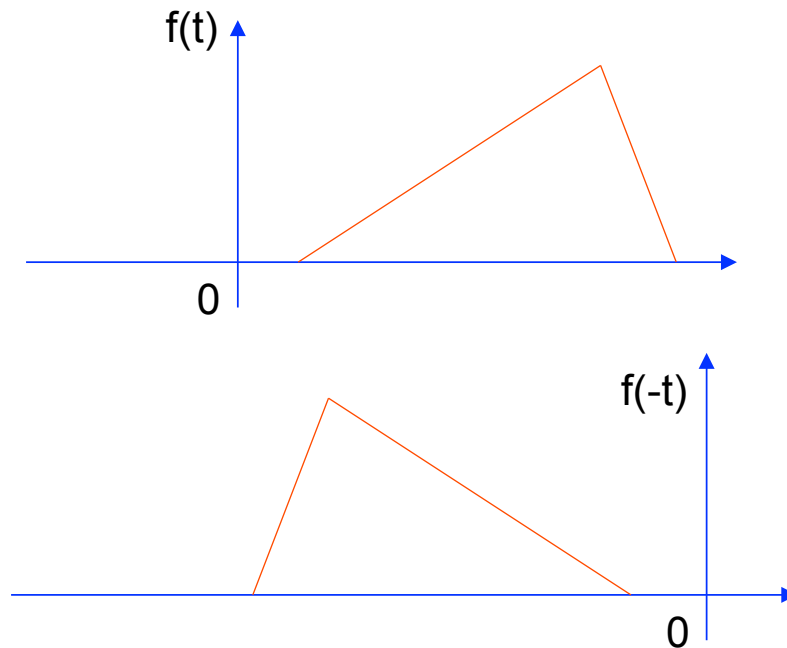$$\varphi(t) = f\left(\frac{t}{a}\right) \rightarrow \text{ dilated (or expanded) version}$$

Viceversa for $a < 1$
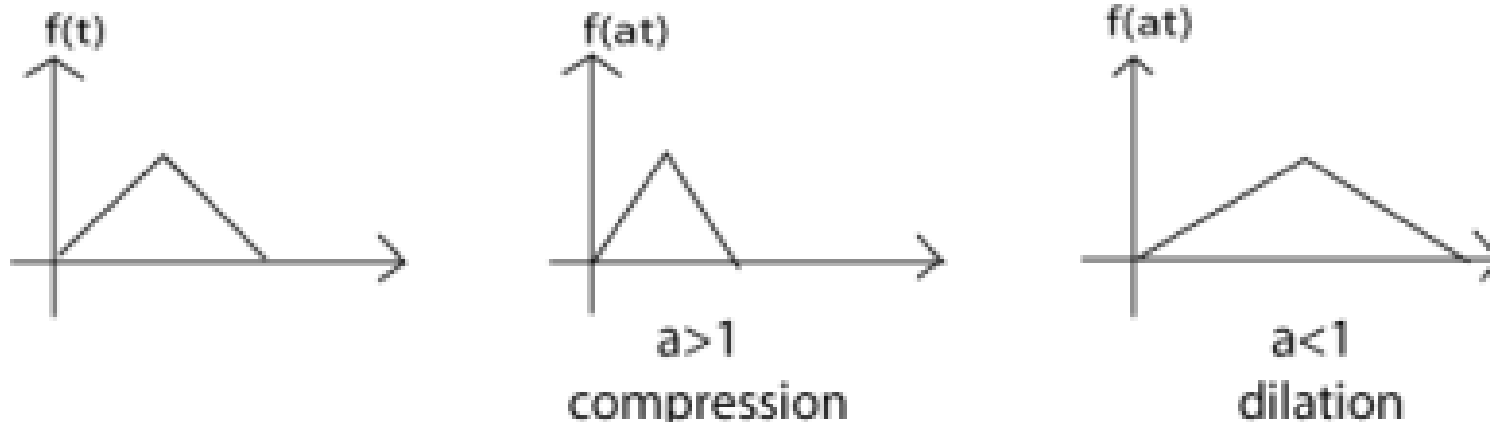
Useful signal operations: shifting, scaling, inversion

- (Time) inversion: mirror image of f(t) about the vertical axis

$$\varphi(t) = f(-t)$$

Useful signal operations: shifting, scaling, inversion

- Combined operations: $f(t) \rightarrow f(at-b)$

- Two possible sequences of operations

1. *Time shift f(t) by to obtain f(t-b). Now time scale the shifted signal f(t-b) by a to obtain f(at-b).*

2. *Time scale f(t) by a to obtain f(at). Now time shift f(at) by b/a to obtain f(at-b).*

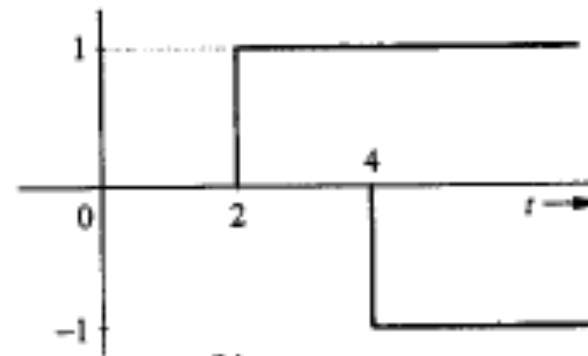   - *Note that you have to replace t by (t-b/a) to obtain f(at-b) from f(at) when replacing t by the translated argument (namely t-b/a))*

Useful functions

- Unit step function
  - Useful for representing causal signals

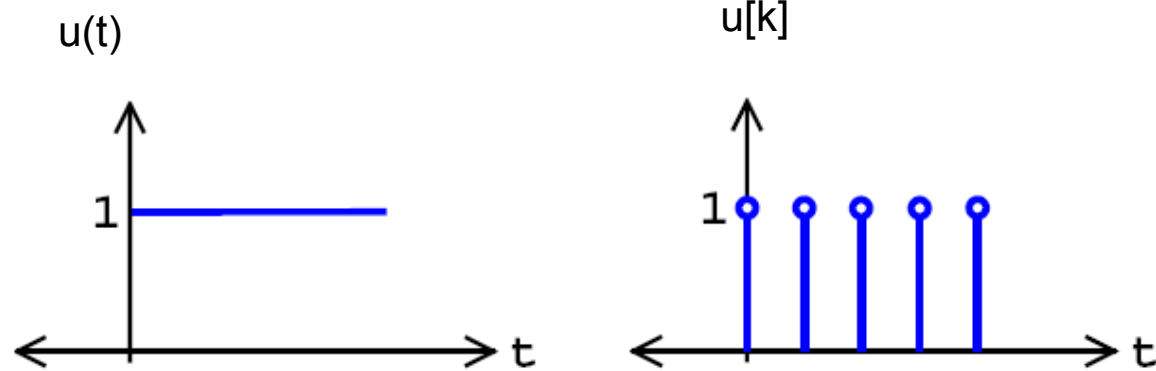$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$
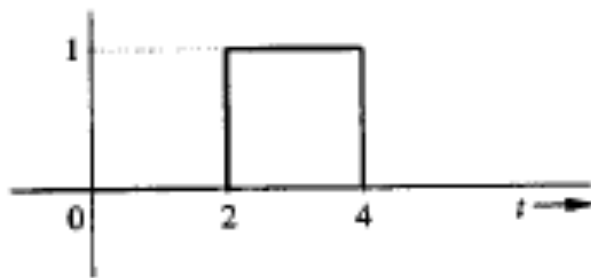
Fig. 1.15 Representation of a rectangular pulse by step functions.

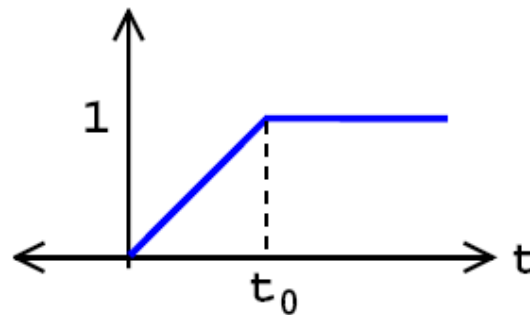$$f(t) = u(t-2) - u(t-4)$$

Useful functions

- Continuous and discrete time unit step functions

u(t)

u[k]

Useful functions

- Ramp function (continuous time)

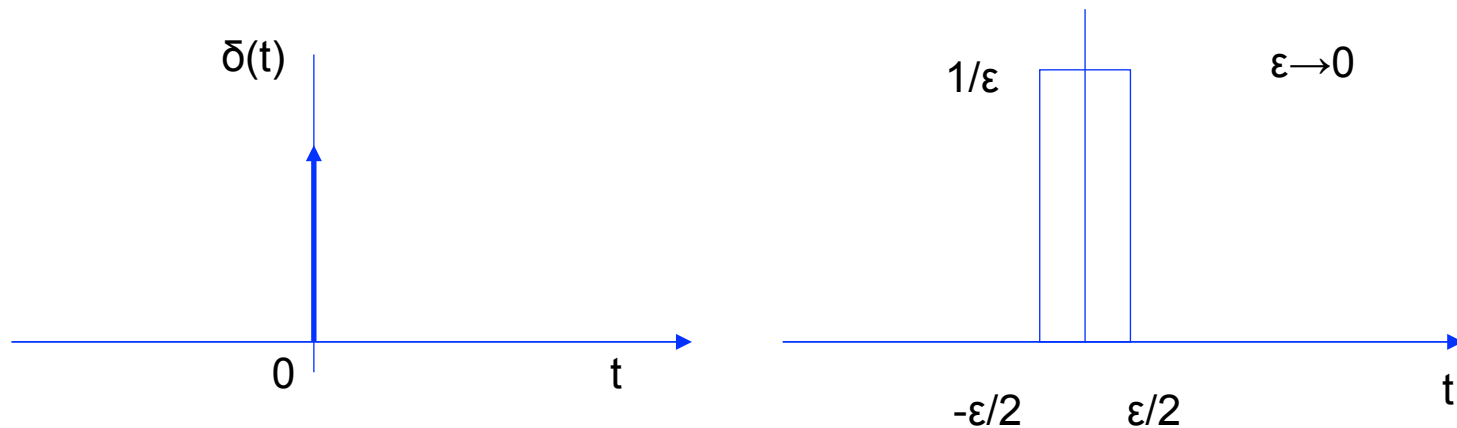$$r\left(t\right) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0} \text{ if } 0 \leq t \leq t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

Useful functions

- Unit impulse function

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

δ(t)

1/ε   ε→0

0   t

-ε/2   ε/2   t

Properties of the unit impulse function

- Multiplication of a function by impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

- Sampling property of the unit function

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)\,dt = \int_{-\infty}^{+\infty} \phi(0)\delta(t)\,dt = \phi(0)\int_{-\infty}^{+\infty} \delta(t)\,dt = \phi(0)$$

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t-T)\,dt = \phi(T)$$

  - The area under the curve obtained by the product of the unit impulse function shifted by T and $\varphi(t)$ is the value of the function $\varphi(t)$ for t=T

Properties of the unit impulse function

- The unit step function is the integral of the unit impulse function

$$\frac{du}{dt} = \delta(t)$$
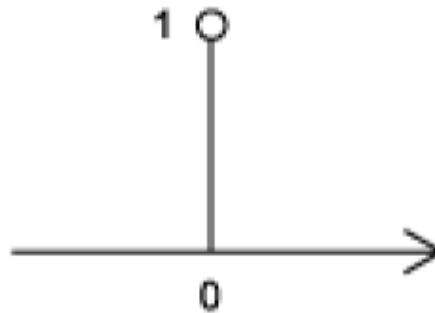
$$\int_{-\infty}^{t} \delta(t)\, dt = u(t)$$

    – Thus

$$\int_{-\infty}^{t} \delta(t)\, dt = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Properties of the unit impulse function

- Discrete time impulse function

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$

Useful functions

- Continuous time complex exponential

$$f(t) = Ae^{j\omega t}$$

- Euler's relations

$$Ae^{j\omega t} = A\cos(\omega t) + j(A\sin(\omega t))$$

$$\cos(\omega t) = \frac{e^{jwt} + e^{-(jwt)}}{2}$$

$$\sin(\omega t) = \frac{e^{jwt} - e^{-(jwt)}}{2j}$$

$$e^{jwt} = \cos(\omega t) + j\sin(\omega t)$$

- Discrete time complex exponential
  - k=nT

$$f[n] = Be^{snT}$$
$$= Be^{j\omega nT}$$

Useful functions

- Exponential function e^{st}
  - Generalization of the function $e^{j\omega t}$

$$s = \sigma + j\omega$$

Therefore

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos \omega t + j \sin \omega t) \qquad (1.30a)$$
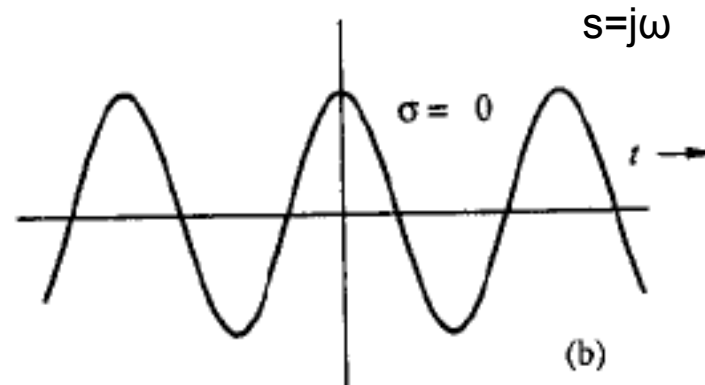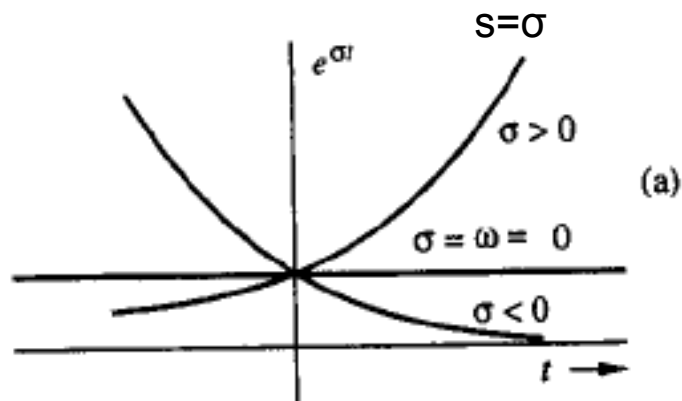
If $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^*t} = e^{\sigma-j\omega} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(\cos \omega t - j \sin \omega t) \qquad (1.30b)$$
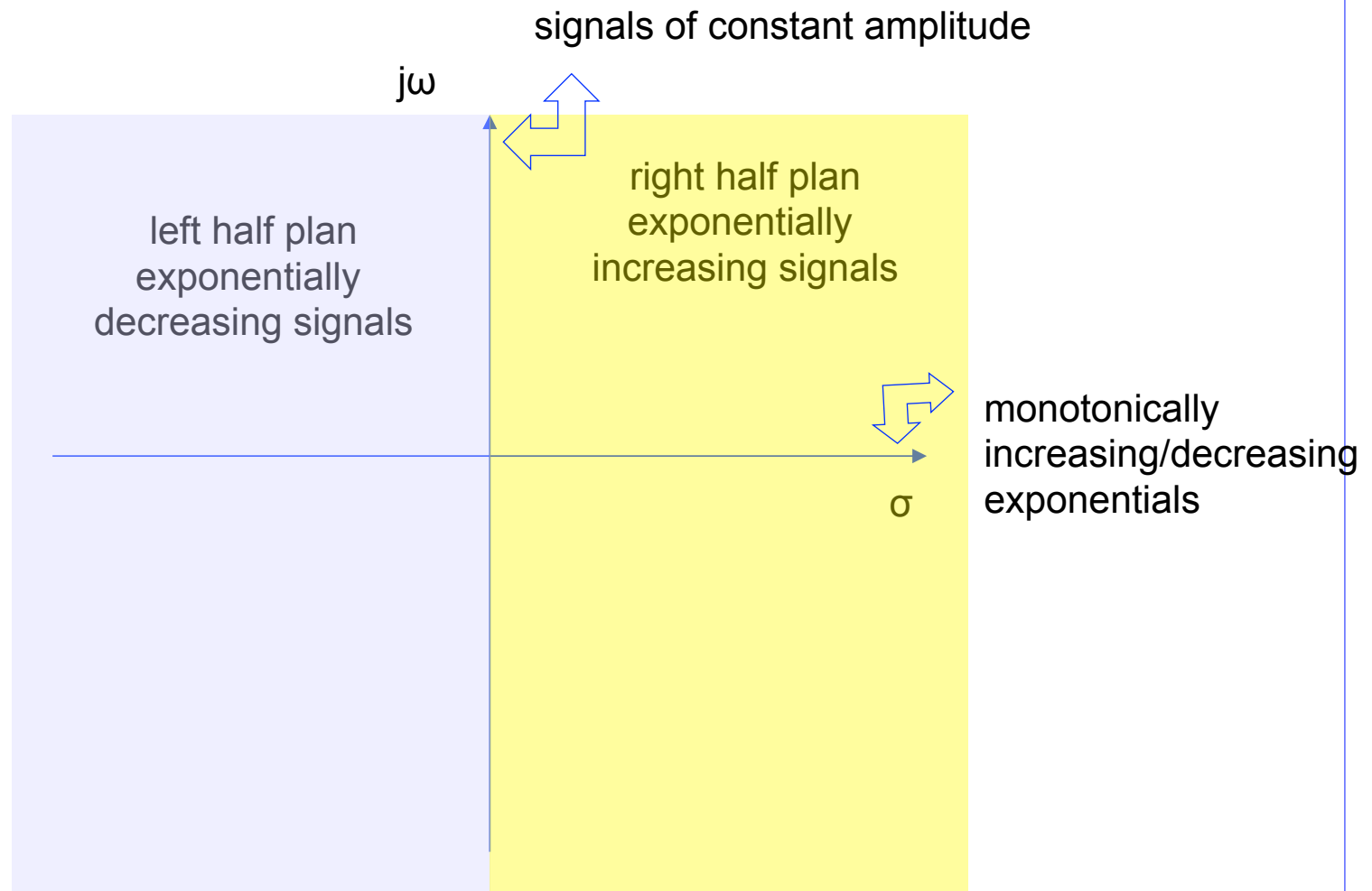
and

$$e^{\sigma t}\cos \omega t = \frac{1}{2}(e^{st} + e^{s^*t}) \qquad (1.30c)$$

The exponential function

Complex frequency plan

signals of constant amplitude

$j\omega$

left half plan
exponentially
decreasing signals

right half plan
exponentially
increasing signals

σ

monotonically
increasing/decreasing
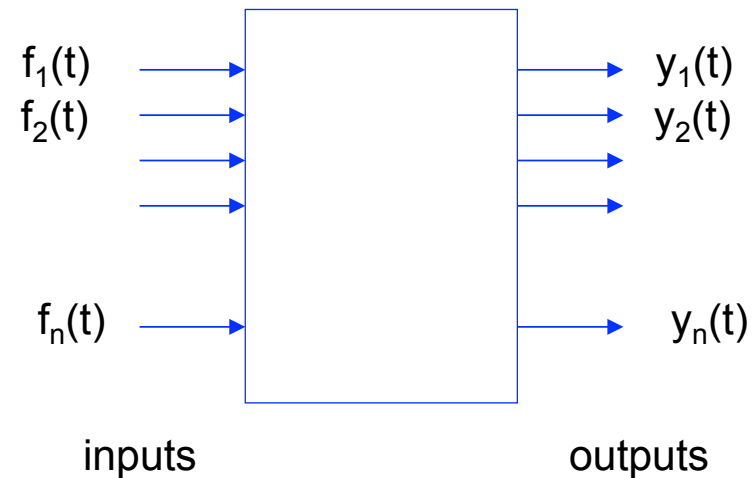exponentials
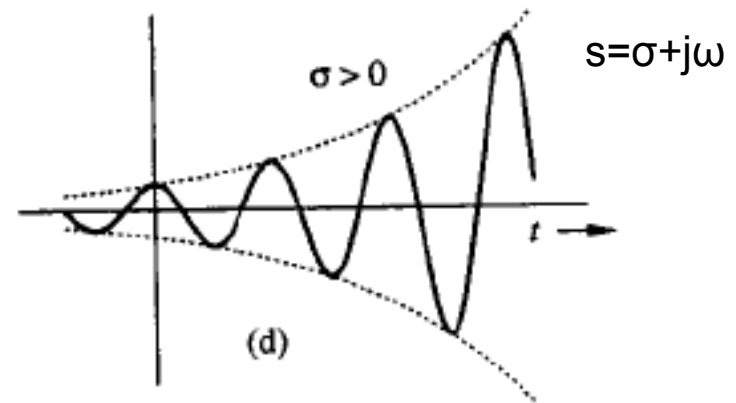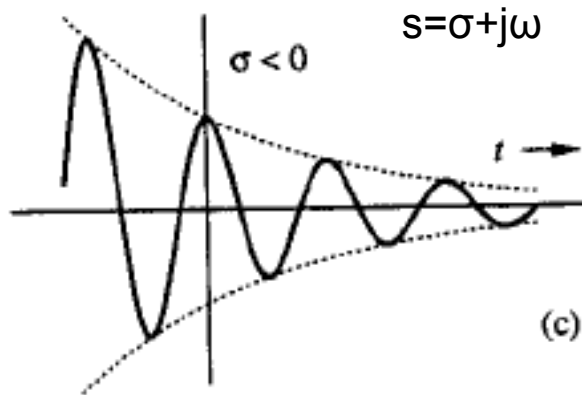
# Basics of Linear Systems

2D Linear Systems

Systems

- A system is characterized by
  - inputs
  - outputs
  - rules of operation (mathematical model of the system)

$f_1(t)$ → [ ] → $y_1(t)$
$f_2(t)$ → → $y_2(t)$

$f_n(t)$ → → $y_n(t)$

inputs   outputs

Systems

- Study of systems: mathematical modeling, analysis, design

    - Analysis: how to determine the system output given the input and the system mathematical model

    - design or synthesis: how to design a system that will produce the desired set of outputs for given inputs

- SISO: single input single output - MIMO: multiple input multiple output

$f_1(t)$ $y_1(t)$

inputs   outputs

$f_1(t)$
$f_2(t)$

$f_n(t)$

$y_1(t)$
$y_2(t)$

$y_n(t)$

inputs   outputs
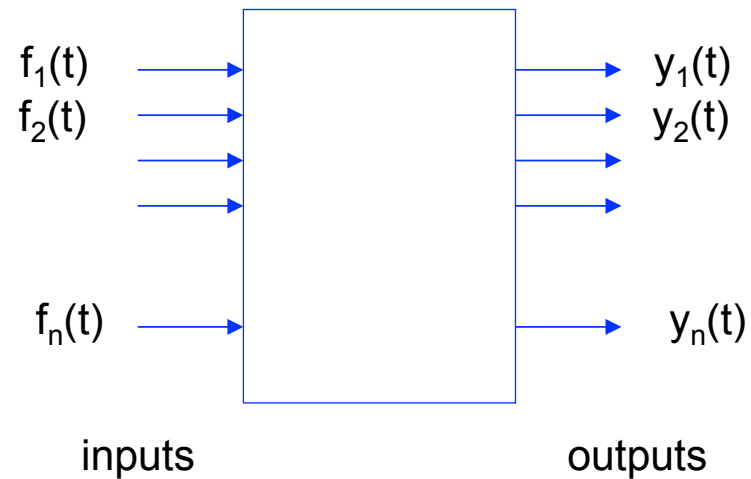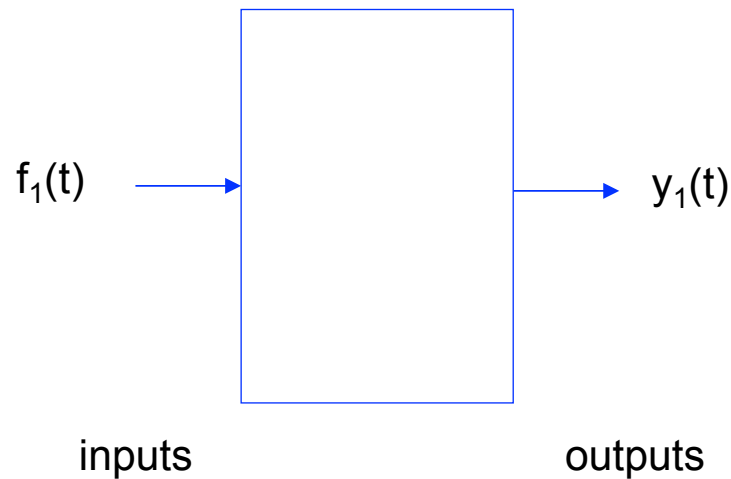
Response of a linear system

- Total response = Zero-input response + Zero-state response

  - The output of a system for t≥0 is the result of two independent causes: the initial conditions of the system (or system state) at t=0 and the input f(t) for t≥0.

  - Because of linearity, the total response is the sum of the responses due to those two causes

  - The zero-input response is only due to the initial conditions and the zero-state response is only due to the input signal

  - This is called decomposition property

- Real systems are *locally* linear

  - Respond linearly to small signals and non-linearly to large signals

causal, linear

causal, non linear

locally linear around f_0
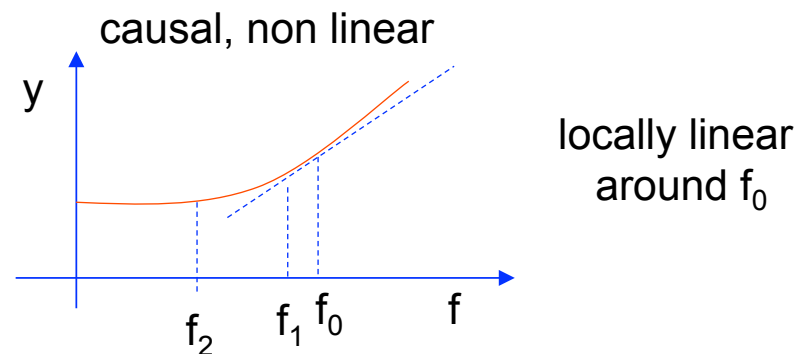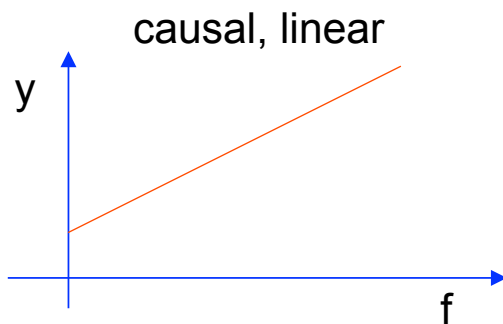
Review: Linear Systems

- We define a system as a unit that converts an input function into an output function

$$f(x) \longrightarrow \boxed{\begin{array}{c} \text{System} \\ H \end{array}} \longrightarrow g(x)$$
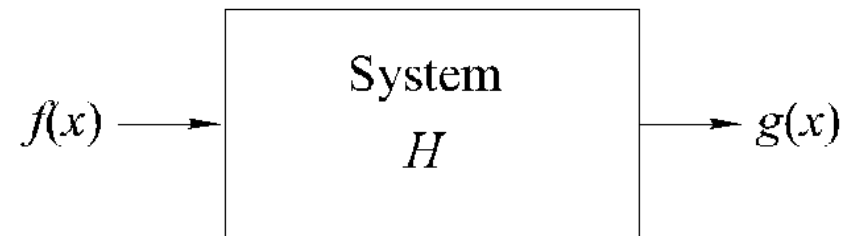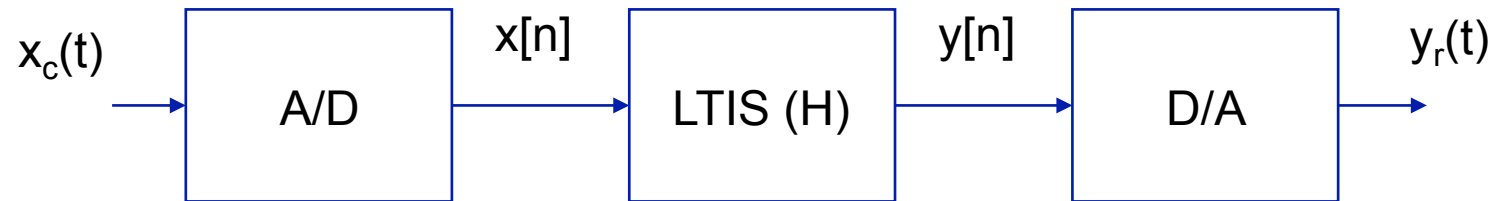
$$g(x) = H[f(x)]$$

Independent System operator or Transfer function variable

Linear Time Invariant Discrete Time Systems

$x_c(t)$ → [A/D] → $x[n]$ → [LTIS (H)] → $y[n]$ → [D/A] → $y_r(t)$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y_r(j\Omega) = H(j\Omega)X_c(j\Omega)$$

$$H(j\Omega) = \begin{cases} H(j\Omega) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

IF	THEN
• The input signal is bandlimited • The Nyquist condition for sampling is met • The digital system is linear and time invariant	The overall continuous time system is equivalent to a LTIS whose frequency response is H.

Overview of Linear Systems

- Let

$$g_i(x) = H[f_i(x)]$$

where $f_i(x)$ is an arbitrary input in the class of all inputs $\{f(x)\}$, and $g_i(x)$ is the corresponding output.

- If

$$H[\alpha_i f_i(x) + \alpha_j f_j(x)] = a_i H[f_i(x)] + a_j H[f_{ji}(x)]$$

$$= a_i g_i(x) + a_j g_j(x)$$

Then the system *H* is called a *linear system*.

- A linear system has the properties of *additivity* and *homogeneity*.

Linear Systems

- The system H is called *shift invariant* if

$$g_i(x) = H[f_i(x)] \text{ implies that } g_i(x + x_0) = H[f_i(x + x_0)]$$

for all $f_i(x) \in \{f(x)\}$ and for all x_0.

- This means that offsetting the independent variable of the input by x_0 causes the same offset in the independent variable of the output. Hence, the input-output relationship remains the same.

Linear Systems

- The operator *H* is said to be *causal*, and hence the system described by *H* is a *causal system*, if there is no output before there is an input. In other words,

$$f(x) = 0 \text{ for } x < x_0 \text{ implies that } g(x) = H[f(x)] = 0 \text{ for } x < x_0.$$

- A linear system *H* is said to be *stable* if its response to any bounded input is bounded. That is, if

$$|f(x)| < K \text{ implies that } |g(x)| < cK$$

where *K* and *c* are constants.

Linear Systems

- A *unit impulse function*, denoted $\delta(a)$, is *defined* by the expression

$$\int_{-\infty}^{\infty} f(\alpha)\delta(x-\alpha)d\alpha = f(x).$$

$\delta(a)$ $\delta(x\text{-}a)$

a

x
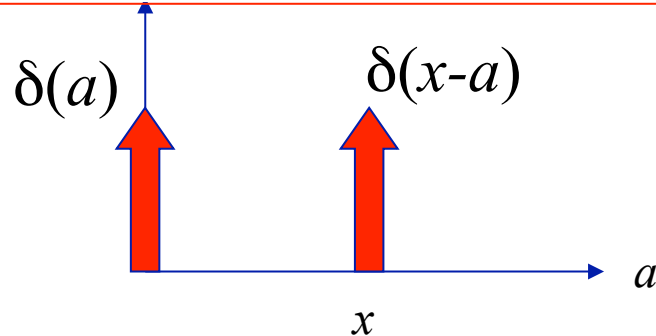
- The response of a system to a unit impulse function is called the *impulse response* of the system

$$h(x) = H[\delta(x)]$$

Linear Systems

- If *H* is a linear shift-invariant system, then we can find its response to any input signal f(x) as follows:

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x - \alpha)d\alpha.$$

70

- This expression is called the *convolution integral*. It states that the response of a linear, fixed-parameter system is completely characterized by the convolution of the input with the system impulse response.

Linear Systems

- Convolution of two functions of a continuous variable is defined as

$$f(x) * h(x) = \int_{-\infty}^{\infty} f(\alpha)h(x - \alpha)d\alpha$$

- In the discrete case

$$f[n] * h[n] = \sum_{m=-\infty}^{\infty} f[m]h[n - m]$$

Linear Systems

- In the 2D discrete case

$$f[n_1, n_2] * h[n_1, n_2] = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} f[m_1, m_2] h[n_1 - m_1, n_2 - m_2]$$

$h[n_1, n_2]$ is a linear filter.

Illustration of the folding, displacement, and multiplication steps needed to perform two-dimensional convolution

Matrix perspective

i	h	g
f	e	d
c	b	a

Step 1

c	b	a
f	e	d
i	h	g

a	b	c
d	e	f
g	h	i

Step 2
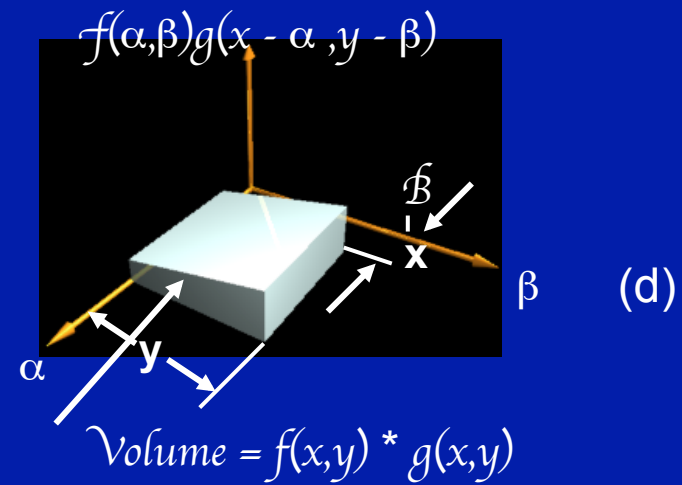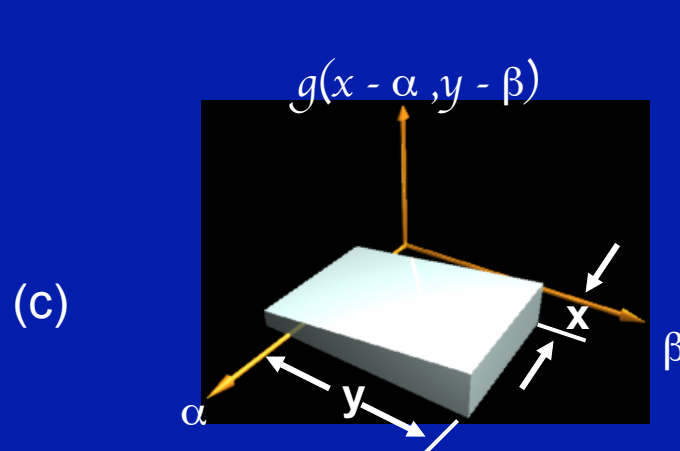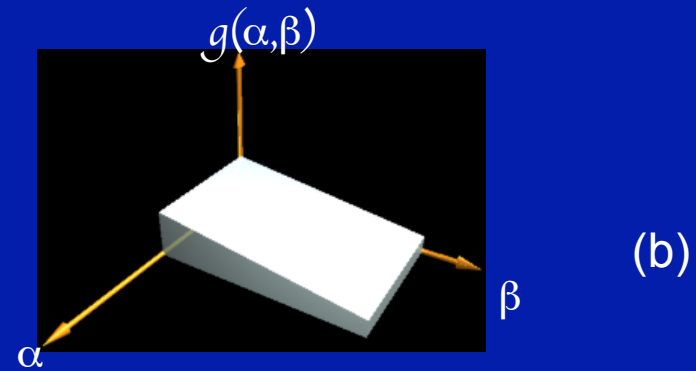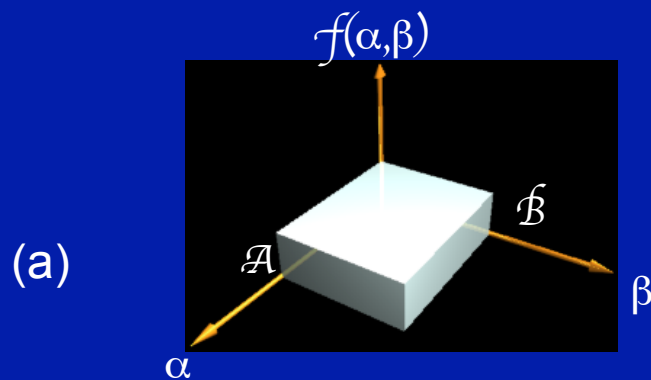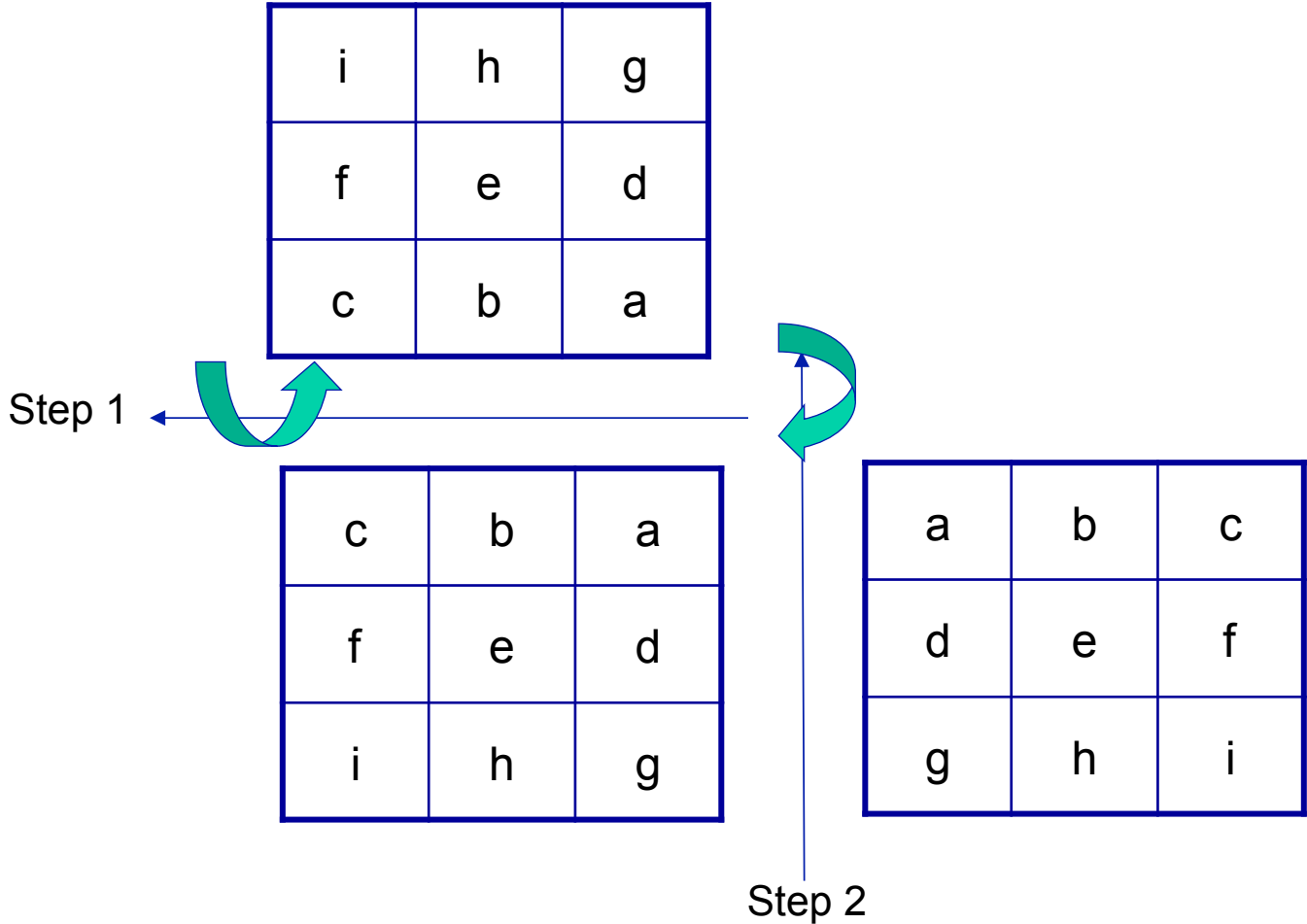
Convolution Example

h

1	-1	-1
1	2	-1
1	1	1

$$f[n_1, n_2] ** h[n_1, n_2] = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f[m_1, m_2] h[n_1 - m_1, n_2 - m_2]$$

Rotate

1	1	1
-1	2	1
-1	-1	1

f

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2
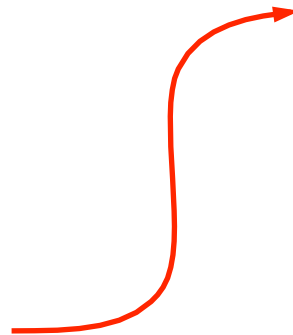
From C. Rasmussen, U. of Delaware

Convolution Example

h

f

f*h

Convolution Example

h

Convolution Example

f   f*h

Convolution Example
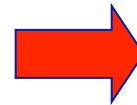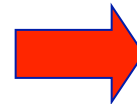

f

f*h

Convolution Example

h

f

f*h

Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

h

2	2	2	3
-2	2	3	3
-2	-2	1	2
1	3	2	2

f

5	4	4	-2
9	6		

f*h