$\Gamma \vdash \varphi(x) \Rightarrow \Gamma \vdash \forall x \varphi(x)$ if $x \notin FV(\psi)$ for all $\psi \in \Gamma$

$\Gamma \vdash \forall x \varphi(x) \Rightarrow \Gamma \vdash \varphi(t)$ if t is free for x in φ.

$\Gamma \vdash \varphi(x) \Leftrightarrow$

$\exists \underset{\varphi(x)}{\mathcal{D}} \quad hp(\mathcal{D}) \subseteq \Gamma$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi(x)\end{matrix}}{\forall x \varphi(x)} \Rightarrow \Gamma \vdash \forall x \varphi(x)$$

$$\forall x(\alpha\to\beta)\to(\forall x\alpha\to\forall x\beta)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\alpha\to\beta)]^2}{\alpha\to\beta}\qquad\dfrac{[\forall x\alpha]^1}{\alpha}}{\beta}}{\forall x\beta}}{\forall x\alpha\to\forall x\beta}\,1}{\forall x(\alpha\to\beta)\to(\forall x\alpha\to\forall x\beta)}\,2$$

$$(\forall x\, \alpha) \to (\neg \underbrace{\forall x \neg \alpha}_{(\forall x \neg \alpha) \to \bot})$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{[\forall x\, \alpha]^{2}}{\alpha} \qquad \cfrac{[\forall x \neg \alpha]^{1}}{\neg \alpha}}{\bot}\,1}{\neg(\forall x \neg \alpha)}\,2}{\forall x\, \alpha \to \neg \forall x \neg \alpha}
$$

```
int f (int x ){
  return 1*2* ... *x; }
```

```
int f (int x){
  return x*f(x-1);}
```

$$\frac{\neg(\alpha \to \beta)}{\alpha}$$

$$\vdash \neg(\alpha \to \beta) \to \alpha$$

$$\vdash (\alpha \to \beta) \leftrightarrow (\neg\alpha \lor \beta)$$

$$\vdash (\delta \to \varepsilon) \leftrightarrow (\neg\varepsilon \to \neg\delta)$$

$$\vdash \neg(\gamma \lor \omega) \leftrightarrow (\neg\gamma \land \neg\omega)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{[\alpha]^1 \quad [\neg\alpha]^2}{\bot}}{\beta}}{\alpha \to \beta} \qquad [\neg(\alpha \to \beta)]^3}{\cfrac{\cfrac{\bot}{\alpha}\,2}{\neg(\alpha \to \beta) \to \alpha}\,3}$$

$$\dfrac{\dfrac{\neg(\alpha\to\beta)}{\quad}}{\alpha}$$

$$\dfrac{\dfrac{\dfrac{\neg(\alpha\to\beta) \qquad \dfrac{\overset{\mathcal{D}_5}{(\neg\alpha\vee\beta)\to(\alpha\to\beta)} \qquad \overset{\mathcal{D}_6}{((\neg\alpha\vee\beta)\to(\alpha\to\beta))\to(\neg(\alpha\to\beta)\to\neg(\neg\alpha\vee\beta))}}{\neg(\alpha\to\beta)\to\neg(\neg\alpha\vee\beta)}}{\neg(\neg\alpha\vee\beta)} \qquad \overset{\mathcal{D}_2}{\neg(\neg\alpha\vee\beta)\to(\neg\neg\alpha\wedge\neg\beta)}}{\dfrac{\neg\neg\alpha\wedge\neg\beta}{\neg\neg\alpha}} \qquad \overset{\mathcal{D}_3}{\neg\neg\alpha\to\alpha}}{\alpha}$$

$$\cancel{\dfrac{\alpha \lor \beta}{\alpha}}$$

$$\dfrac{\neg(\alpha \to \beta)}{\alpha}$$ NON SI UTIL.

$$\cancel{\dfrac{\alpha \qquad \bot}{\alpha \to \bot}}$$

(iii) $\vdash \forall x \varphi(x) \rightarrow \forall z \varphi(z)$ if z does not occur in $\varphi(x)$,

$$\dfrac{\dfrac{\forall x \varphi}{\varphi[z/x]}}{\forall z(\varphi[z/x])}$$

$$\dfrac{\dfrac{\forall x\, \varphi(x)}{\varphi(z)}}{\forall z\, \varphi(z)}$$

$$\forall x(\alpha \vee \beta) \to ((\forall x\, \alpha) \vee (\forall x\, \beta))$$

$v \quad w$

α è vero solo in v

β è vero solo in w

$$\models (\forall x\, \alpha \wedge \forall x\, \beta) \leftrightarrow \forall x (\alpha \wedge \beta)$$

$$\Longleftrightarrow$$

$$\forall \mathfrak{A}, \rho \quad \underbrace{\mathfrak{A}, \rho \models (\forall x\, \alpha \wedge \forall x\, \beta) \leftrightarrow \forall x (\alpha \wedge \beta)}$$

$$\mathfrak{A}, \rho \models \forall x\, \alpha \wedge \forall x\, \beta \Leftrightarrow$$

$$\mathfrak{A}, \rho \models \forall x\, \alpha \ \& \ \mathfrak{A}, \rho \models \forall x\, \beta \Leftrightarrow$$

$$(\forall a\ \mathfrak{A}, \rho[x \mapsto a] \models \alpha) \ \& \ (\forall a\ \mathfrak{A}, \rho[x \mapsto a] \models \beta) \Leftrightarrow$$

$$\forall a \left[\mathfrak{A}, \rho[x \mapsto a] \models \alpha \ \& \ \mathfrak{A}, \rho[x \mapsto a] \models \beta \right] \Leftrightarrow$$

$$\forall a \left[\mathfrak{A}, \rho[x \mapsto a] \models \alpha \wedge \beta \right] \Leftrightarrow \mathfrak{A}, \rho \models \forall x (\alpha \wedge \beta)$$

$$\models \forall x\, \alpha \to \exists x\, \alpha$$

$\boxed{\because\ \forall \mathfrak{A}\ \forall \rho\, (\mathfrak{A}, \rho \not\models \forall x \alpha \text{ or } \mathfrak{A}, \rho \models \exists x \alpha)}$

$$\Longleftrightarrow$$

$$\forall \mathfrak{A}\ \forall \rho\, (\mathfrak{A}, \rho \models \forall x \alpha \to \exists x \alpha)$$

$$\Longleftrightarrow$$

$$\forall \mathfrak{A}, \forall \rho\, (\mathfrak{A}, \rho \models \forall x \alpha \Rightarrow \mathfrak{A}, \rho \models \exists x \alpha)$$

$$\Longleftrightarrow$$

$$\forall \mathfrak{A}, \forall \rho\, \big([\forall a \in |\mathfrak{A}|\ \mathfrak{A}, \rho[x \mapsto a] \models \alpha] \Rightarrow [\exists a \in |\mathfrak{A}|\ \mathfrak{A}, \rho[x \mapsto a] \models \alpha] \big)$$

$\checkmark$ ok

$\models \exists x (\alpha \vee \beta) \leftrightarrow (\exists x \alpha \vee \exists x \beta)$ facile

$\models \forall x \alpha \rightarrow \alpha[t/x]$ difficile

$\not\models \forall x (\alpha \vee \beta) \rightarrow (\forall x \alpha \vee \forall x \beta)$ medio

$\models \exists x \alpha \rightarrow \exists x (\alpha \vee \beta)$

$\not\models \exists x \alpha \rightarrow \forall x \alpha$

$\models \forall x (\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta)$

$\not\models (\forall x \alpha \rightarrow \forall x \beta) \rightarrow \forall x (\alpha \rightarrow \beta)$

$$(\alpha \to \beta) \models \neg\beta \to \neg\alpha$$

$$\Leftrightarrow$$

$$\forall v\left([\alpha \to \beta]_v = 1 \Rightarrow [\neg\beta \to \neg\alpha]_v = 1\right)$$

$$\Leftrightarrow$$

$$\forall v\left(\{[\alpha]_v = 0 \text{ OR } [\beta]_v = 1\}\right.$$

$$\Rightarrow$$

$$\left.\{\underbrace{[\neg\beta]_v = 0}_{[\beta]_v = 1} \text{ OR } \underbrace{[\neg\alpha]_v = 1}_{[\alpha]_v = 0}\}\right)$$

$$\underbrace{(\alpha \to \beta) \vdash \neg\beta \to \neg\alpha}$$

$$\boxed{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \beta \quad [\alpha]^1}{\beta} \quad [\neg\beta]^2}{\bot}}{\neg\alpha}\,1}{\neg\beta \to \neg\alpha}}$$

$$\forall v\left(\{[\alpha]_v = 0 \text{ OR } [\beta]_v = 1\} \Rightarrow \{[\beta]_v = 1 \text{ OR } [\alpha]_v = 0\}\right)$$

$$\neg(\alpha \vee \beta) \vdash \neg\alpha$$

$$\dfrac{\dfrac{\dfrac{[\alpha]^1}{\alpha \vee \beta} \quad \neg(\alpha \vee \beta)}{\bot}}{\neg\alpha}\,1$$

$$\neg(\alpha \vee \beta) \vDash \neg\alpha$$

$$\forall v \left([\neg(\alpha \vee \beta)]_v = 1 \Rightarrow [\neg\alpha]_v = 1 \right) \Leftrightarrow$$

$$\forall v \left([\alpha \vee \beta]_v = 0 \Rightarrow [\alpha]_v = 0 \right) \Leftrightarrow$$

$$\forall \left(\left([\alpha]_v = 0 \;\&\; [\beta]_v = 0 \right) \Rightarrow [\alpha]_v = 0 \right) \quad \text{ok}$$

$$(\alpha \to \beta) \vdash (\beta \to c) \to (\alpha \to c)$$

$\alpha \to \beta \vdash \neg\beta \to \neg\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \beta \quad [\alpha]^1}{\beta} \quad [\beta \to c]^2}{c}}{\alpha \to c}}{(\beta \to c) \to (\alpha \to c)}$$

$$\alpha \to (\beta \wedge \neg \beta) \vdash \neg \alpha$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha]^1 \quad \alpha \to (\beta \wedge \neg \beta)}{\beta \wedge \neg \beta}}{\beta} \qquad \dfrac{\dfrac{[\alpha]^1 \quad \alpha \to (\beta \wedge \neg \beta)}{\beta \wedge \neg \beta}}{\neg \beta}}{\bot}}{\neg \alpha}$$

$$\vdash \beta \wedge \neg \beta \to \bot$$

$$\dfrac{\dfrac{\dfrac{[\alpha]^2 \quad \alpha \to (\beta \wedge \neg \beta)}{\beta \wedge \neg \beta} \qquad \dfrac{\dfrac{\dfrac{[\beta \wedge \neg \beta]^1}{\beta} \quad \dfrac{[\beta \wedge \neg \beta]^1}{\neg \beta}}{\bot}}{\beta \wedge \neg \beta \to \bot}}{\bot}}{\neg \alpha}$$

$$(\alpha \to \alpha) \vdash (\bot \to \bot)$$

$$\frac{[\bot]^1}{\bot \to \bot}$$

$$\vdash \bot \to \bot$$

$$\Gamma \vdash \alpha \quad \Rightarrow \quad \Gamma \cup \Delta \vdash \alpha$$

$$\vdash \alpha \quad \Rightarrow \quad \Gamma \vdash \alpha$$

$$(\alpha \wedge \beta), \neg\alpha \vdash \beta$$

$$\dfrac{\begin{matrix}[\neg\alpha] \\ \vdots \\ \bot\end{matrix}}{\alpha} \qquad \dfrac{\bot}{\alpha}$$

$$\dfrac{\alpha \wedge \beta}{\beta} \qquad \dfrac{\dfrac{\alpha \wedge \beta}{\alpha} \quad \neg\alpha}{\dfrac{\bot}{\beta}}$$

$\downarrow$

$$(\alpha \wedge \beta), \delta \vdash \beta$$

$\neg \forall x \varphi \rightarrow \exists x \neg \varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\varphi]^1}{\exists x \neg\varphi} \quad [\neg\exists x\neg\varphi]^2}{\bot}}{\varphi}\,1}{\forall x \varphi} \quad \neg\forall x \varphi}{\dfrac{\bot}{\exists x \neg \varphi}}$$

$$\cancel{\dfrac{\exists x \varphi \quad \dfrac{[\varphi]}{\forall x \varphi}}{\forall x \varphi}}$$

φ

$$(\forall x P \to \bot) \to \exists x (P \to \bot)$$

$$\neg \forall x P \to \exists x \neg P$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]^1[\neg P]^2}{\bot}}{\neg P}}{\exists x(\quad)} \qquad [\neg \exists x(P \to Q)]^3}{\bot}}{P}\,2}{\forall x P} \qquad (\forall x P) \to Q}{Q}}{P \to Q}}{\exists x (P \to Q)} \qquad [\neg \exists x (P \to Q)]^3}{\bot}}{\exists x (P \to Q)}$$

$\alpha \equiv \left(\forall x (P(x) \vee Q(x))\right) \to \left(\forall x P(x) \vee \forall x Q x\right)$ $\quad \Gamma \models A \Leftrightarrow \Gamma \vdash A$

$\mathfrak{M} = \langle \{a, b\}, \underset{\cup}{p}, \underset{\cup}{q} \rangle \qquad P = \{a\} \quad Q = \{b\}$

$\mathfrak{M} \not\models \alpha \Leftrightarrow \exists \sigma \quad \mathfrak{M}, \sigma \not\models \alpha$

$\mathfrak{M}, \sigma \not\models \left(\forall x (P(x) \vee Q(x))\right) \to \left(\forall x P(x) \vee \forall x Q x\right)$

$\Leftrightarrow$

$\underbrace{\mathfrak{M}, \sigma \models \forall x (P(x) \vee Q(x)}_{(1)}) \;\&\; \underbrace{\mathfrak{M}, \sigma \not\models \forall x P(x) \vee \forall x Q(x)}_{(2)}$

1)

$$\Longleftrightarrow$$

$$\underbrace{\mathfrak{M},\sigma \models \forall x(P(x) \vee Q(x))}_{(1)} \ \& \ \underbrace{\mathfrak{M},\sigma \not\models \forall x P(x) \vee \forall x\, Q(x)}_{(2)}$$

1)

$$\mathfrak{M},\sigma \models \forall x\,(P(x) \vee Q(x)) \Longleftrightarrow$$

$$\forall z \in \{a,b\} \quad \mathfrak{M},\ \sigma[x \mapsto z] \models P(x) \vee Q(x)$$

$$\Longleftrightarrow$$

$$\forall z \in \{a,b\} \left(\sigma[x \mapsto z] \models P(x) \ \text{ OR } \ \sigma[x \mapsto z] \models Q(x) \right)$$

$$\Longleftrightarrow$$

$$\forall z \in \{a,b\} \left([\![x]\!]_{\sigma[x \mapsto z]} \in p \ \text{ OR } \ [\![x]\!]_{\sigma[x \mapsto z]} \in q \right)$$

$$\forall z \in \{a,b\} \left([\![x]\!]_{\sigma[x \mapsto z]} \in p \text{ OR } [\![x]\!]_{\sigma[x \mapsto z]} \in q \right)$$

$$\Leftrightarrow$$

$$\left([\![x]\!]_{\sigma[x \mapsto a]} \in p \text{ OR } [\![x]\!]_{\sigma[x \mapsto a]} \in q \right)$$

$$\&$$

$$\left([\![x]\!]_{\sigma[x \mapsto b]} \in p \text{ OR } [\![x]\!]_{\sigma[x \mapsto b]} \in q \right)$$

$$\Leftrightarrow$$

$$(a \in p \text{ OR } a \in q) \ \& \ (b \in p \text{ OR } b \in q)$$

$$\sigma \not\models \forall x\, P(x) \lor \forall x\, q(x)$$

$$\Longleftrightarrow$$

$$\sigma \not\models \forall x\, P(x) \quad \& \quad \sigma \not\models \forall x\, q(x)$$

$$\Longleftrightarrow$$

$$\left(\exists z \in \{a,b\}\ \ \sigma[x \mapsto z] \not\models P(x)\right) \qquad \sigma[m \mapsto n]$$

$$\&$$

$$\left(\exists z \in \{a,b\}\ \ \sigma[x \mapsto z] \not\models Q(x)\right)$$

$$\Longleftrightarrow$$

$$\left(\exists z \in \{a,b\}\ \ [\![x]\!]_{\sigma[x \mapsto z]} \notin p\right) \& \left(\underbrace{\exists z \in \{a,b\}\ \ [\![x]\!]_{\sigma[x \mapsto z]}}_{z=a} \notin q\right)$$

SI BASTA PRENDERE $z = b$

$b \notin p$ $\qquad$ $a \notin q$

$$
\dfrac{\exists x \varphi(x) \qquad \begin{array}{c} [\varphi] \\ \mathcal{D} \\ \psi \end{array}}{\psi}\ \exists E
$$

$x \notin FV(hp\mathcal{D} - \{\varphi\}) \cup FV(\psi)$

$$
\dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{array}{c} [\varphi_1] \\ | \\ \psi \end{array} \qquad \begin{array}{c} [\varphi_2] \\ | \\ \psi \end{array}}{\psi}
$$

$t_1, t_2 \ldots, t_k \ldots$ $\qquad \exists x \varphi(x) \approx \varphi(t_1) \vee \varphi(t_2) \vee \varphi(t_3) \vee \ldots$

$$
\dfrac{\overbrace{\varphi(t_1) \vee \varphi(t_2) \vee \varphi(t_3) \vee \ldots}^{\exists x \varphi(x)} \qquad \begin{array}{c} \varphi(t_1) \\ | \\ \psi \end{array} \qquad \begin{array}{c} \varphi(t_2) \\ | \\ \psi \end{array} \qquad \begin{array}{c} \varphi(t_3) \\ | \\ \psi \end{array} \qquad \ldots}{\psi}
$$

$t_1, t_2, \ldots, t_k, \ldots \qquad \exists x\, \varphi(x) \approx \varphi(t_1) \vee \varphi(t_2) \vee \varphi(t_3) \vee \ldots$

$$\dfrac{\overbrace{\varphi(t_1) \vee \varphi(t_2) \vee \varphi(t_3) \vee \ldots}^{\exists x\, \varphi(x)} \qquad \boxed{\begin{array}{ccccc} \varphi(t_1) & \varphi(t_2) & \varphi(t_3) & & \\ \mathcal{D}_1 \Big| & \mathcal{D}_2 \Big| & \mathcal{D}_3 \Big| & \ldots & \\ \psi & \psi & \psi & & \end{array}}}{\psi}$$

$$\begin{array}{c} \varphi(x) \\ \Big| \mathcal{D}^* \\ \psi \end{array}$$

$FV($

$x \notin hp(\mathcal{D}^*))$

$\varphi(x)$

$x \notin FV(\psi)$

$\vdash \forall x \varphi(x) \to \exists x \varphi(x)$

$\exists I$

$$\frac{\varphi(t)}{\exists x \varphi(x)}$$

$$\dfrac{\dfrac{\dfrac{[\forall x \varphi(x)]^{*}}{\varphi(x)}}{\exists x \varphi(x)}}{\forall x \varphi(x) \to \exists x \varphi(x)}$$

$\neg, \forall, \exists$

$\wedge, \vee$

$\to$

$$\dfrac{\varphi(t)}{\exists x\,\varphi(x)}$$

$$\exists x\,(\alpha(x)\vee\beta(x))\to(\exists x\,\alpha(x)\vee\exists x\,\beta(x))$$

$$\dfrac{\exists x\,\varphi(x) \qquad \begin{array}{c}[\varphi(x)]\\ \vdots \\ \psi\end{array}}{\psi}$$

$$\dfrac{\dfrac{[\exists x\,(\alpha(x)\vee\beta(x))]^{\square} \qquad \dfrac{[\alpha(x)\vee\beta(x)]^{\circ} \qquad \dfrac{\dfrac{[\alpha(x)]^{*}}{\exists x\,\alpha(x)}}{\exists x\,\alpha(x)\vee\exists x\,\beta(x)} \qquad \dfrac{\dfrac{[\beta(x)]^{*}}{\exists x\,\beta(x)}}{\exists x\,\alpha(x)\vee\exists x\,\beta(x)}}{\exists x\,\alpha(x)\vee\exists x\,\beta(x)}\,{*}}{\exists x\,\alpha(x)\vee\exists x\,\beta(x)}\,{\circ}}{\exists x\,(\alpha(x)\vee\beta(x))\to(\exists x\,\alpha(x)\vee\exists x\,\beta(x))}\,{\square}$$

$$\exists x\,(\alpha(x)\wedge\beta(x)) \rightarrow (\exists y\,\alpha(y)\wedge\exists z\,\beta(z))$$

$$\dfrac{\dfrac{[\exists x\,(\overbrace{\alpha(x)\wedge\beta(x)}^{\psi(x)})]^{*} \qquad \dfrac{\dfrac{\dfrac{[\overbrace{\alpha(x)\wedge\beta(x)}^{\psi(x)}]^{0}}{\alpha(x)}}{\exists y\,\alpha(y)} \qquad \dfrac{\dfrac{[\overbrace{\alpha(x)\wedge\beta(x)}^{\psi(x)}]^{0}}{\beta(x)}}{\exists z\,\beta(z)}}{\exists y\,\alpha(y)\wedge\exists z\,\beta(z)}}{\exists y\,\alpha(y)\wedge\exists z\,\beta(z)}\,{}^{0}}{\exists x(\alpha(x)\wedge\beta(x)) \rightarrow (\exists y\,\alpha(y)\wedge\exists z\,\beta(z))}\,{}^{*}$$

$$
\dfrac{\dfrac{\dfrac{[\exists x\,\varphi(x)]^{\star} \qquad \dfrac{[\varphi(x)]^{\circ} \qquad \dfrac{[\forall x\,\neg\varphi(x)]^{*}}{\neg\varphi(x)}}{\bot}}{\bot}\,\circ}{\neg\forall x\,\neg\varphi(x)}\,*}{\exists x\,\varphi(x) \rightarrow \neg\forall x\,\neg\varphi(x)}\,\star
$$

$$(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$$

$$\dfrac{\dfrac{\dfrac{\boxed{\begin{matrix}\mathcal{D}\\ \alpha \vee \neg\alpha\end{matrix}} \qquad \dfrac{[\alpha]^{\circ}\ [\alpha\to\beta]^{\star}}{\beta} \qquad \dfrac{[\neg\alpha]^{\circ}\ [\neg\alpha\to\beta]^{\ast}}{\beta}}{\beta}\ \circ}{(\neg\alpha\to\beta)\to\beta}\ \ast}{(\alpha\to\beta)\to((\neg\alpha\to\beta)\to\beta)}\ \star$$

$$(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$$

$$\neg\beta \to ((\alpha \to \beta) \to \neg\alpha) \qquad \neg\beta \to ((\neg\alpha \to \beta) \to \alpha)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\beta]^{*} \qquad \dfrac{[\alpha \to \beta]^{\circ} \quad [\alpha]^{\dagger}}{\beta}}{\bot}}{\neg\alpha}\dagger}{(\alpha \to \beta) \to \neg\alpha}}{\neg\beta \to ((\alpha \to \beta) \to \neg\alpha)}* \qquad\qquad \dfrac{\dfrac{\neg\beta \qquad \dfrac{\neg\alpha \to \beta \quad [\neg\alpha]^{\dagger}}{\beta}}{\bot}}{\alpha}$$

$$(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$$

$$
\dfrac{
\dfrac{
\boxed{\dfrac{\dfrac{[\neg\beta]^{\circ} \quad \dfrac{[\alpha \to \beta]^{\boxtimes} \quad [\alpha]^{\dagger}}{\beta}}{\bot}}{\neg\alpha}\,\dagger}
\qquad
\boxed{\dfrac{\dfrac{[\neg\beta]^{\circ} \quad \dfrac{[\neg\alpha \to \beta]^{*} \quad [\neg\alpha]^{\dagger}}{\beta}}{\bot}}{\alpha}\,\dagger}
}{
\dfrac{\bot}{\beta}\,\circ
}
}{
\dfrac{(\neg\alpha \to \beta) \to \beta}{(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)}\,\boxtimes
}
$$

$\mathcal{R} = \langle \mathbb{R}, <, =, +, *, 0, 1 \rangle$

$\langle 2; 2, 2; 2 \rangle$

$<, =, +, *, 0, 1$ simboli: I-ord

PER OGNI NUMERO REALE X
ESISTE UN NUMERO REALE Y
TALE CHE Y È MAGGIORE DI X

$\alpha \quad \mathcal{R} \models \alpha$

$\forall x \exists y . x < y \equiv \alpha \quad \mathcal{R} \models \alpha$

$\mathfrak{M} = \langle A, R_1 - R_n, f_1, -, f_k, c_1 - \rangle$

$\mathcal{N} = \langle \mathbb{N}, <, \sigma, +, 0 \rangle$

$\langle 2; 1, 2; 1 \rangle$

α I NUMERI NATURALI NON HANNO MASSIMO

$\alpha \equiv \sim (\exists x \forall y (y < x))$

$P \subseteq A$ $\quad$ $P(x)$

P è formalizzata da una formula α t.c. $FV(\alpha) = z$

t.c. $a \in P$ ovvero $P(a) \Leftrightarrow$

$\{ a \mid \rho[z \mapsto a] \models \alpha \} = P$

$P \subseteq A \qquad P(x)$

P è formalizzata da una formula α t.c. $FV(\alpha) = z$

t.c. $a \in P$ ovvero $P(a) \iff$

$$\{a \mid \rho[z \mapsto a] \models \alpha\} = P$$

x è un numero pari

$$\alpha(x) \equiv \exists y . x = 2 \cdot y$$

$$P = \{a \in \mathbb{N} \mid \rho[x \mapsto a] \models x = 2 \cdot y\} \iff$$

$$P = \{a \in \mathbb{N} \mid a \text{ è pari}\}$$

$\mathcal{N} = \langle \mathbb{N}, <, +, \bullet, \sigma, 0 \rangle$

$\alpha(x)$ x è un numero dispari

$\neg(z=0)$

$$\exists z \left(x = (\underbrace{\sigma(\sigma(0))}_{2} \bullet z) + \underbrace{\sigma(0)}_{1} \right)$$

$\alpha(x_1, \ldots x_n)$ $Fv(\alpha) = \{x_1, \ldots, x_n\}$

$\alpha(x, y)$ x è un multiplo di y

$\exists z (x = z \cdot y)$ $\exists z (z \neq 0 \wedge x = z \cdot y)$

$\exists z (x = \bullet(z, y))$

$$1 \equiv \sigma(0)$$

$$PR(x) \equiv \neg(x=0) \wedge \big(\neg(x=1) \wedge$$
$$\wedge \ \forall y \forall z \ (x = y \cdot z \rightarrow (x=y \ \vee \ x=z))\big)$$

L'INSIEME DEI NUMERI PRIMI NON HA MASSIMO

α I NUMERI NATURALI NON HANNO MASSIMO

$$\alpha \equiv \sim(\exists x \forall y \, (y<x))$$

$$\sim(\exists x \, (Pr(x) \wedge \forall y \, (PR(y) \rightarrow (y<x))))$$

OGNI MULTIPLO DI 4 È PARI

PARI(x)

$$4 \equiv \sigma(\sigma(\sigma(\sigma(0)))) \ldots$$

M4(z) z è MULT. DI 4

$$M4(z) \equiv \exists y \, (y \neq 0 \land z = 4 \cdot y)$$

$$\forall x \, (M4(x) \to \text{PARI}(x))$$

$\langle A, \leq \rangle$ t.c. $\# A < \aleph_0$

A ha almeno un massimale

$\rightarrow \mathrm{MASM}(x) \leftarrow$

$\mathrm{MASM}(x) \equiv \forall y \, (x \leq y \rightarrow x = y)$

$\exists z \ \mathrm{MASM}(z)$

$$\Rightarrow \langle \mathcal{P}(\mathbb{N}), \subseteq, \qquad \emptyset \rangle \Leftarrow$$

ESISTE $x \subseteq \mathbb{N}$ t.c. $|x| = 1$

$$C1(x) \equiv |x| = 1$$

$$z \subseteq \{a\} \iff \begin{cases} z = \emptyset \\ z = \{a\} \end{cases}$$

$$C1(x) \equiv \forall z \left(z \subseteq x \rightarrow (z = x \vee z = \emptyset) \right)$$

∀x(A→B)⊢¬(∀xA)∨∀xB

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(A \to B)}{A \to B} \qquad \dfrac{[\forall x A]^{1}}{A}}{B}}{\forall x B}}{\neg(\forall x A) \vee \forall x B} \qquad [\neg(\neg \forall x A \vee \forall x B)]^{2}}{\perp}\,1}{\neg \forall x A}}{\neg \forall x A \vee \forall x B} \qquad [\neg(\neg \forall x A \vee \forall x B)]^{2}}{\dfrac{\perp}{\neg \forall x A \vee \forall x B}\,2}
$$

$aRb \ \& \ bRc \Rightarrow aRc$

$a = 0 \Rightarrow b = 0 \Rightarrow c = 0$

$a \neq 0$

- $b = 0 \Rightarrow c = 0 \quad aR0$
- $b \neq 0$
  - $c = 0 \quad aR0$
  - $c \neq 0 \quad a \leq b \ \& \ b \leq c \Rightarrow a \leq c \Rightarrow aRc$